To Grid or Not To Grid

Atomic Methods for Sparse Inverse Problems

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EPFL

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LCAV

Atomic Methods for Sparse Inverse Problems

$\in \mathbb{R}^L$ $\in \mathbb{R}^N$ $L \leq N$

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$$

On Grid

Deconvolution

Inpainting

Fourier sampling

Atomic Methods for Sparse **Inverse Problems**

Super resolution

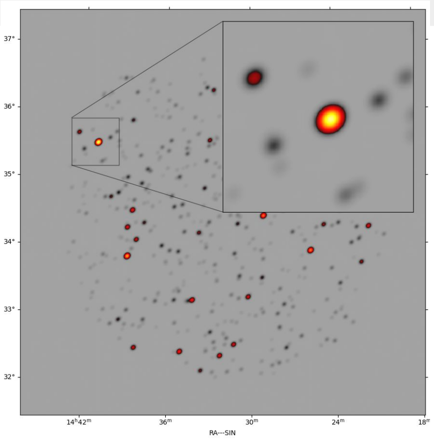
SMLM

$$\mathbf{y} = \mathbf{\Phi}(f) + \mathbf{n}$$

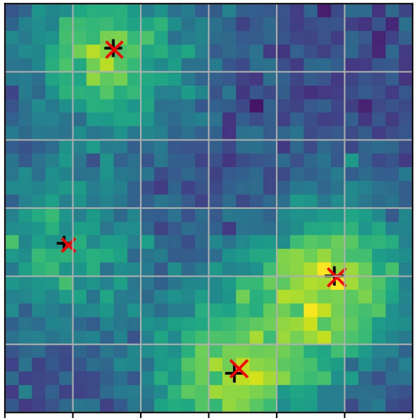
$f : \mathbb{R}^d \to \mathbb{R}$

Off-the-Grid

Catalogue Boötes field[1]



Continuous-Domain Gaussian deconvolution *[Chapter 5]*

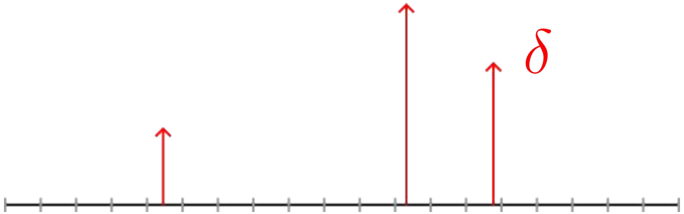


$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$$

$$\mathbf{y} = \boldsymbol{\Phi}(f) + \mathbf{n}$$

Atomic Methods for Sparse Inverse Problems

$\mathbf{x} =$ 

$m =$ 

[1] Williams WL et al., "LOFAR 150-MHz observations of the Boötes field: catalogue and source counts.", *Monthly Notices of the Royal Astronomical Society,* 2016

$$\arg\min_{\mathbf{x}\in\mathbb{R}^N} E(\mathbf{y}, \mathbf{A}\mathbf{x}) + \mathcal{R}(\mathbf{x})$$

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$$

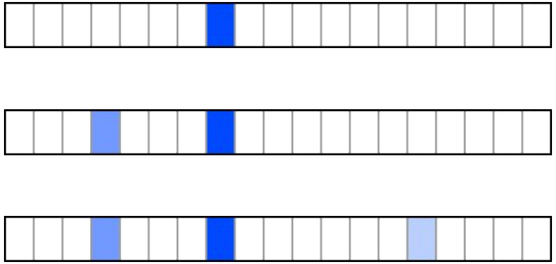
$$\arg\min_{m\in\mathcal{M}(\mathbb{R}^d)} E(\mathbf{y}, \mathbf{\Phi}(m)) + \mathcal{R}(m)$$

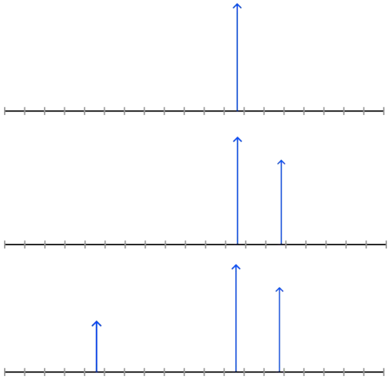
$$\mathbf{y} = \mathbf{\Phi}(f) + \mathbf{n}$$

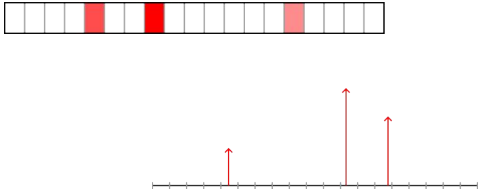
→ Compressed sensing theory

→ Representer theorems

Principled Atomic Methods for Sparse Inverse Problems

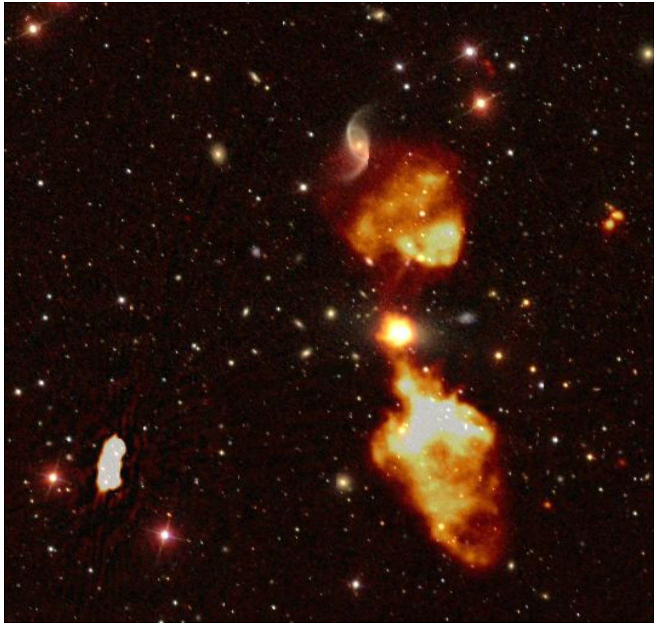






1. **The PolyCLEAN Journey**
    a. Polyatomic Frank-Wolfe for the LASSO
    b. A competitive Imaging Framework
2. Reconstruction beyond the Grid
    a. Another Polyatomic Approach
    b. Decoupling of Composite Sparse-plus-Smooth problems
3. Conclusion

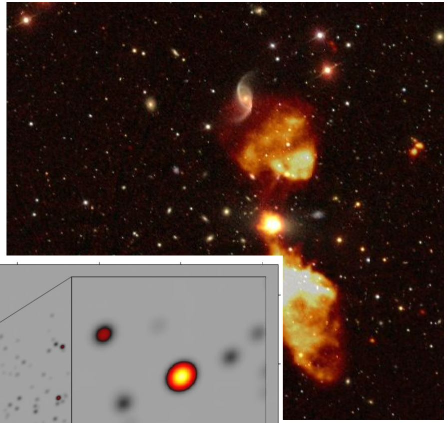
Chapter 7



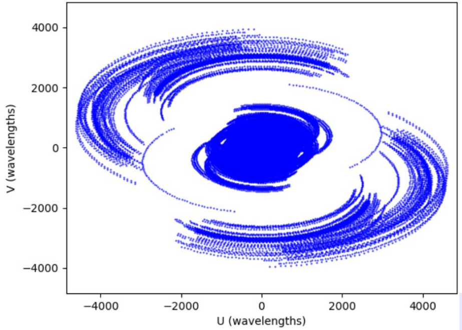
[Credits: C. Tasse and the LOFAR surveys team.]

Radio Interferometric Imaging

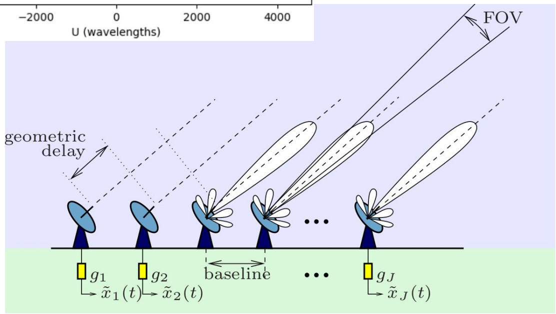
$\mathbf{I}$



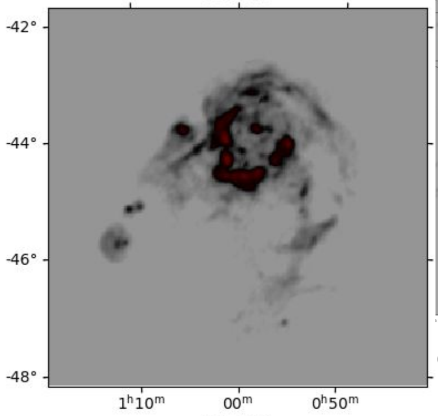
LOFAR baselines
(48 stations)



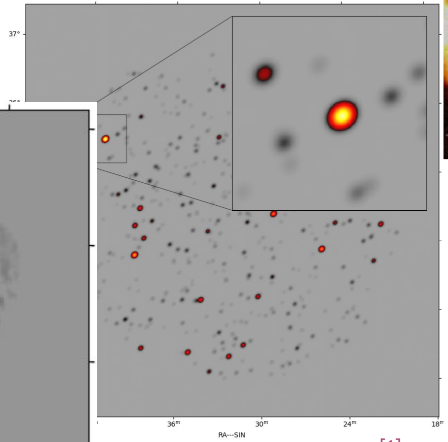
Φ



$$\mathbf{V} = \Phi \mathbf{I} + \varepsilon$$



M31 from RASCIL



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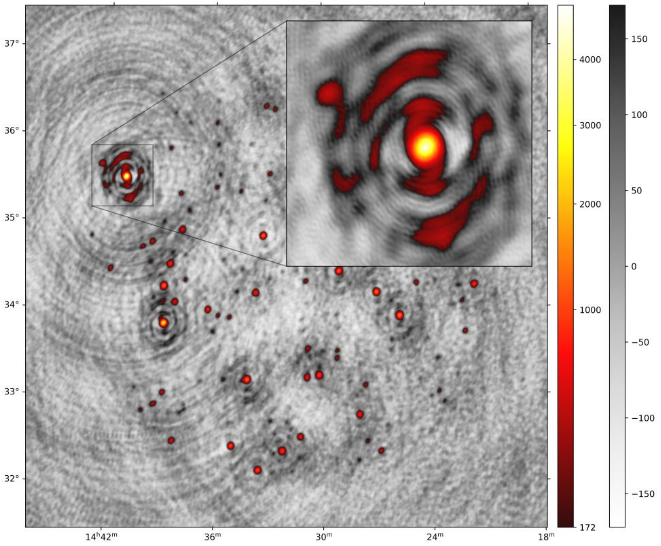
[1] Williams WL et al., “LOFAR 150-MHz observations of the Boötes field: catalogue and source counts.”, *Monthly Notices of the Royal Astronomical Society*, 2016
[2] Van der Veen et al. “Signal Processing for Radio Astronomy”, 2019

[Credits: C. Tasse and the LOFAR surveys team.]

Radio Interferometric Imaging - Dirty Image

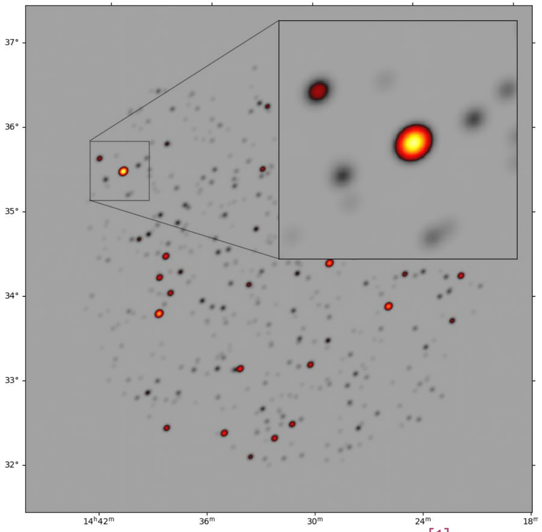
$$\mathbf{V} = \mathbf{\Phi}\mathbf{I} + \varepsilon$$

$$\mathbf{I}_D = \mathbf{\Phi}^*\mathbf{V}$$



Dirty image from observations

$$\mathbf{I}$$



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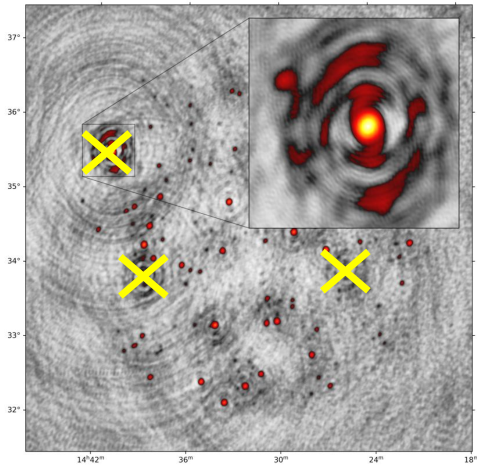
Classical Approaches

The CLEAN family[3]

$\mathbf{I}_D$



✅ Intuitive and simple method, long-developed and fast

❌ Sensitive to stop, objective function unclear, physically impossible artefacts

Optimization-based Methods[4]

$$\arg\min_{\mathbf{I}\in\mathbb{R}^N} \frac{1}{2}\|\mathbf{y} - \mathbf{\Phi}\mathbf{I}\|_2^2 + \mathcal{R}(\mathbf{I})$$

✅ Controlled solutions, versatile priors, excellent results, additional tools

[3] Högbom JA., "Aperture Synthesis with a Non-Regular Distribution of Interferometer Baselines", *Astronomy and Astrophysics Supplement Series*, 1974.

[4] Wiaux Y. et al., "Compressed sensing imaging techniques for radio interferometry", *Monthly Notices of the Royal Astronomical Society*, 2009.

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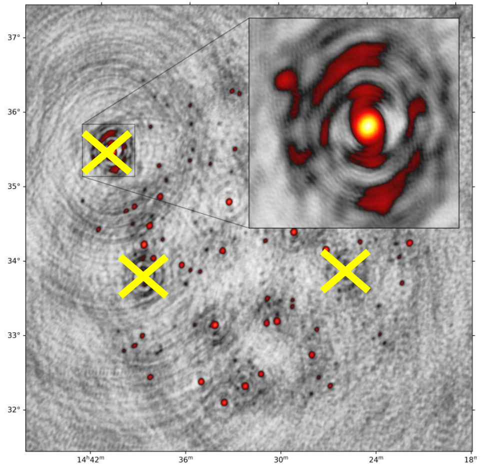

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$$\underset{\mathbf{I}\in\mathbb{R}^N}{\arg\min}\ \frac{1}{2}\|\mathbf{y}-\mathbf{\Phi}\mathbf{I}\|_2^2+\boxed{\lambda\|\mathbf{I}\|_1}$$

LASSO

✅ Controlled solutions, versatile priors, excellent results, additional tools

❌ Numerically heavy, little adoption in the field

[3] Högbom JA., "Aperture Synthesis with a Non-Regular Distribution of Interferometer Baselines", *Astronomy and Astrophysics Supplement Series*, 1974.

[4] Wiaux Y. et al., "Compressed sensing imaging techniques for radio interferometry", *Monthly Notices of the Royal Astronomical Society*, 2009.

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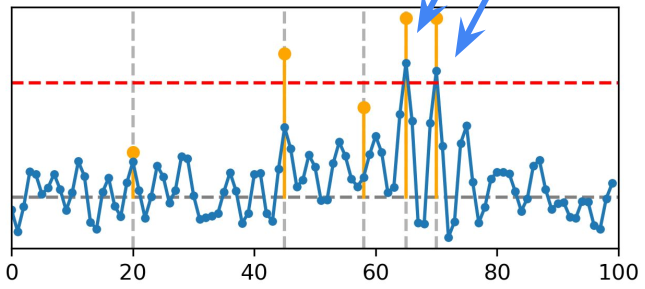
Chapter 3



The Vanilla Frank-Wolfe Algorithm

$$\boxed{\arg\min_{\mathbf{x}\in\mathcal{D}} \mathcal{J}(\mathbf{x})}$$

- $\mathcal{J}$: Convex, differentiable
- $\mathcal{D}$: Convex, bounded domain

Algorithm 1: Vanilla Frank-Wolfe algorithm [5]

Initialize $\mathbf{x}_0 \in \mathcal{D}$

for $k = 1, 2 \cdots$ **do**

1) Find an update direction:

$$\mathbf{s}_k \in \arg\min_{\mathbf{s}\in\mathcal{D}} \mathcal{J}(\mathbf{x}_k) + \langle \nabla \mathcal{J}(\mathbf{x}_k), (\mathbf{s} - \mathbf{x}_k) \rangle$$

2.a) Step size: $\gamma_k \leftarrow 2/(k+2)$

2.b) Reweight:

$$\mathbf{x}_k \leftarrow (1-\gamma_k)\mathbf{x}_k + \gamma_k \mathbf{s}_k = \mathbf{x}_k + \gamma_k(\mathbf{s}_k - \mathbf{x}_k)$$

[5] Frank M, Wolfe P., “An algorithm for quadratic programming”, *Naval Research Logistics Quarterly*, 1956.

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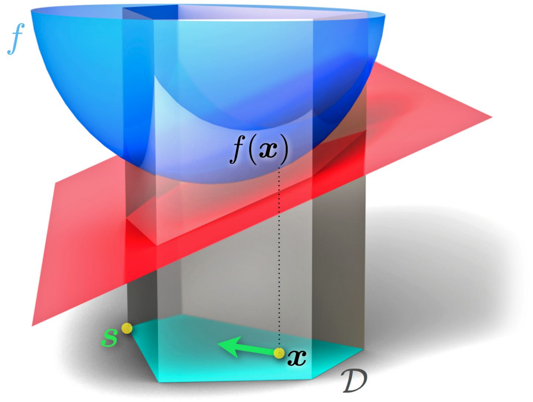

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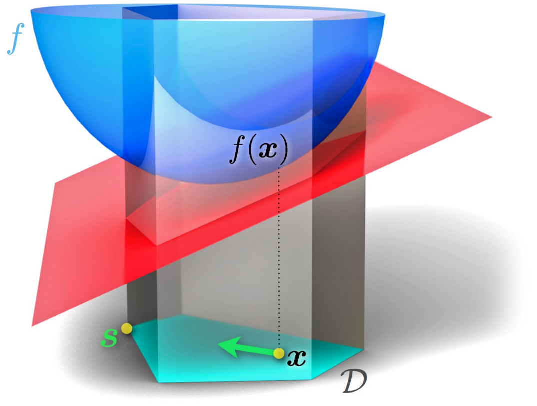


Convergence[6]:

$$\mathcal{J}(\mathbf{x_k}) - \mathcal{J}^* = \mathcal{O}(1/k)$$

[5] Frank M, Wolfe P., "An algorithm for quadratic programming", *Naval Research Logistics Quarterly*, 1956.

[6] Jaggi M., "Revisiting Frank-Wolfe: Projection-Free Sparse Convex Optimization", *Proceedings of the 30th International Conference on Machine Learning, PMLR*, 2013.

Frank-Wolfe for the LASSO

$$\arg\min_{\mathbf{x}\in\mathbb{R}^N} \mathcal{J}(\mathbf{x}) := \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1$$ [7, 8]

Algorithm 2: Vanilla Frank-Wolfe algorithm

Initialize $\mathbf{x}_0 \in \mathcal{D}$

for $k = 1, 2 \cdots$ **do**

1) Find an update direction:

$$\mathbf{s}_k = \mathbf{e}_{i_k} \quad \text{with} \quad i_k = \arg\max_{k\in\{1,\ldots,N\}} |\boldsymbol{\eta}_k|$$

2.a) Step size: $\gamma_k \leftarrow 2/(k+2)$

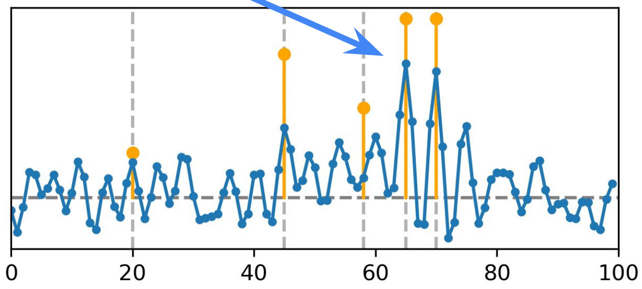
2.b) Reweight:

$$\mathbf{x}_k \leftarrow (1-\gamma_k)\mathbf{x}_k + \gamma_k \mathbf{s}_k = \mathbf{x}_k + \gamma_k(\mathbf{s}_k - \mathbf{x}_k)$$

Canonical basis vector

Empirical dual certificate

$$\boxed{\boldsymbol{\eta}_k = \frac{1}{\lambda}\mathbf{A}^*(\mathbf{y} - \mathbf{A}\mathbf{x}_k)}$$



[7] Denoyelle Q et al. “The sliding Frank–Wolfe algorithm and its application to super-resolution microscopy”, *Inverse Problems*, 2019.
[8] Harchaoui Z. et al., “Conditional gradient algorithms for machine learning”, *NIPS Workshop on Optimization for ML*, 2013.

Our Polyatomic Frank-Wolfe Algorithm

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Algorithm 3: Polyatomic Frank-Wolfe algorithm of quality δ[9]

Initialize $\mathbf{x}_0 \in \mathcal{D}$, $\mathcal{S}_0 = \emptyset$

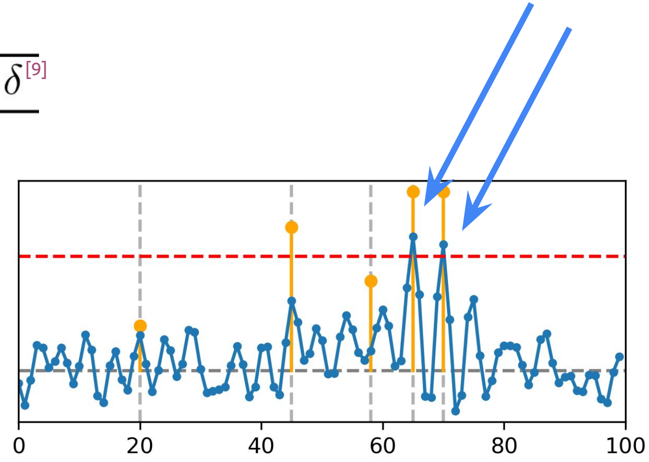
for $k = 1, 2 \cdots$ **do**

1) Find update directions:

$$\mathcal{I}_k \leftarrow \{1 \leq j \leq N : |\boldsymbol{\eta}_k[j]| \geq \|\boldsymbol{\eta}\|_\infty - \delta/k\}$$
$$\mathcal{S}_k \leftarrow \mathcal{S}_{k-1} \cup \mathcal{I}_k$$

2) Reweight:

$$\mathbf{x}_k \leftarrow \arg\min_{\mathrm{Supp}(\mathbf{x})\subset\mathcal{S}_k} \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1$$



[9] **Jarret A**, Fageot J, Simeoni M. "A Fast and Scalable Polyatomic Frank-Wolfe Algorithm for the LASSO", *IEEE Signal Processing Letters*, 2022.

Our Polyatomic Frank-Wolfe Algorithm

$$\underset{\mathbf{x}\in\mathbb{R}^N}{\arg\min}\ \mathcal{J}(\mathbf{x}) := \frac{1}{2}\|\mathbf{y}-\mathbf{Ax}\|_2^2 + \lambda\|\mathbf{x}\|_1$$

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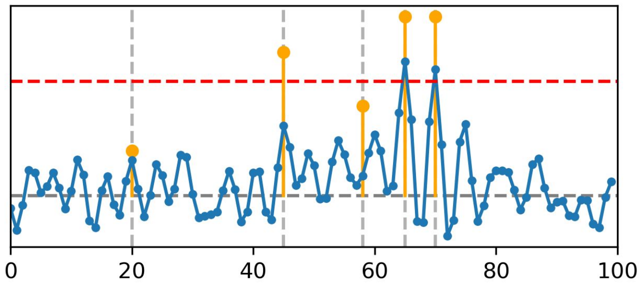

$$\boxed{|\mathcal{S}_k| \ll N}$$

[9] **Jarret A**, Fageot J, Simeoni M. “A Fast and Scalable Polyatomic Frank-Wolfe Algorithm for the LASSO”, *IEEE Signal Processing Letters*, 2022.

Benefits of Polyatomic Frank-Wolfe

- Polyatomic → **Fast**
- Sparse iterates → **Scalable**
- Convergence → **Principled**

$$\mathcal{S}_k \leftarrow \mathcal{S}_{k-1} \cup \mathcal{I}_k$$

$$\mathbf{x}_k = \sum_{i=1}^{N_k} \alpha_i^{[k]} \mathbf{e}_i^{[k]}$$

Theorem 3.2 (Convergence of Polyatomic FW).[9]

$$\mathcal{J}(\mathbf{x}_k) - \mathcal{J}^* \leq \frac{2}{k+2}(C_f + 2\delta)$$

[9] **Jarret A**, Fageot J, Simeoni M. "A Fast and Scalable Polyatomic Frank-Wolfe Algorithm for the LASSO", *IEEE Signal Processing Letters*, 2022.

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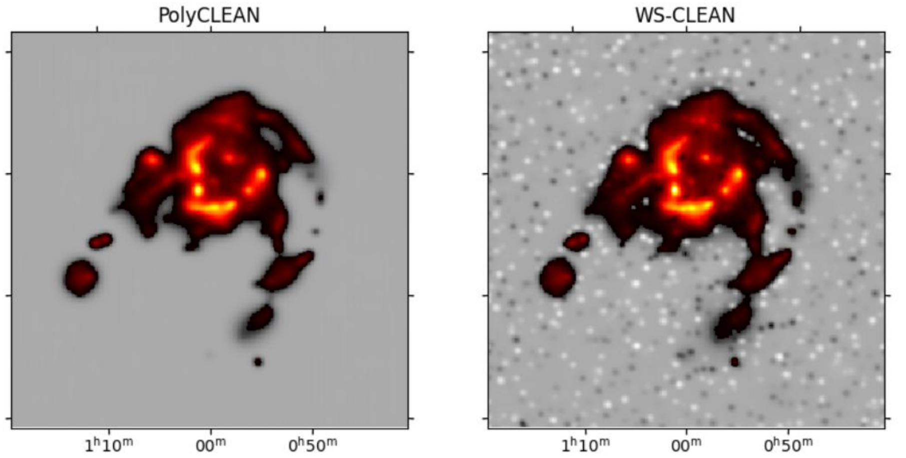
Chapter 8



What's in the name? "PolyCLEAN"

$$\mathbf{V} = \mathbf{\Phi}\mathbf{I} + \varepsilon$$

$$\arg\min_{\mathbf{I}\in\mathbb{R}^N} \frac{1}{2}\|\mathbf{V} - \mathbf{\Phi}\mathbf{I}\|_2^2 + \lambda\|\mathbf{I}\|_1$$

Algorithm 8.1: PolyCLEAN of quality $0 < \delta \leq 1$

Initialize: $\mathbf{I}_0 \leftarrow 0, \mathcal{S}_0 \leftarrow \emptyset, \Delta \leftarrow (1-\delta)\|\mathbf{\Phi}^*\mathbf{V}\|_\infty$

for $k = 0, 1, 2, \ldots$ **do**

Dirty residual: $\boldsymbol{\eta}_k \leftarrow \mathbf{\Phi}^*(\mathbf{V} - \mathbf{\Phi}(\mathbf{I}_k))$

Empirical dual certificate

1.a. Polyatomic exploration:

$$\mathcal{I}_{k+1} = \{1 \leq j \leq N : |\boldsymbol{\eta}_k|_j \geq \|\boldsymbol{\eta}_k\|_\infty - 2\Delta/(k+2)\}$$

1.b. Update active indices:

$$\mathcal{S}_{k+1} \leftarrow \mathcal{S}_k \cup \mathcal{I}_{k+1}$$

2. Update active weights:

$$\mathbf{I}_{k+1} \leftarrow \arg\min_{\mathrm{Supp}(\mathbf{I})\subset\mathcal{S}_{k+1}} \tfrac{1}{2}\|\mathbf{V} - \mathbf{\Phi}\mathbf{I}\|_2^2 + \lambda\|\mathbf{I}\|_1$$

3. Prune atoms:

$$\mathcal{S}_{k+1} \leftarrow \mathrm{Supp}(\mathbf{I}_{k+1})$$

4. Check convergence:

STOP if a stopping criterion is verified.

Output:

Postprocess $\mathbf{I}^{(k)}$ (e.g., convolution with synthetic beam, add residual image)

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Algorithm 8.1: PolyCLEAN of quality $0 < \delta \leq 1$

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$$\mathbf{V} = \boldsymbol{\Phi}\mathbf{I} + \varepsilon$$

$$\underset{\mathbf{I}\in\mathbb{R}^N}{\arg\min}\ \frac{1}{2}\|\mathbf{V} - \boldsymbol{\Phi}\mathbf{I}\|_2^2 + \lambda\|\mathbf{I}\|_1$$

Empirical dual certificate

Polyatomic Frank-Wolfe steps

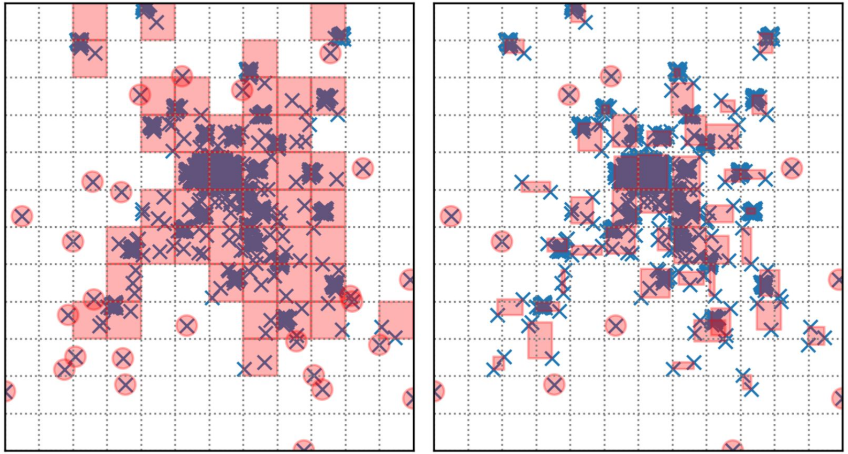
~ Major cycles of CLEAN

Symbiosis with HVOX[10]

Sparsity-aware implementation of the forward operator:

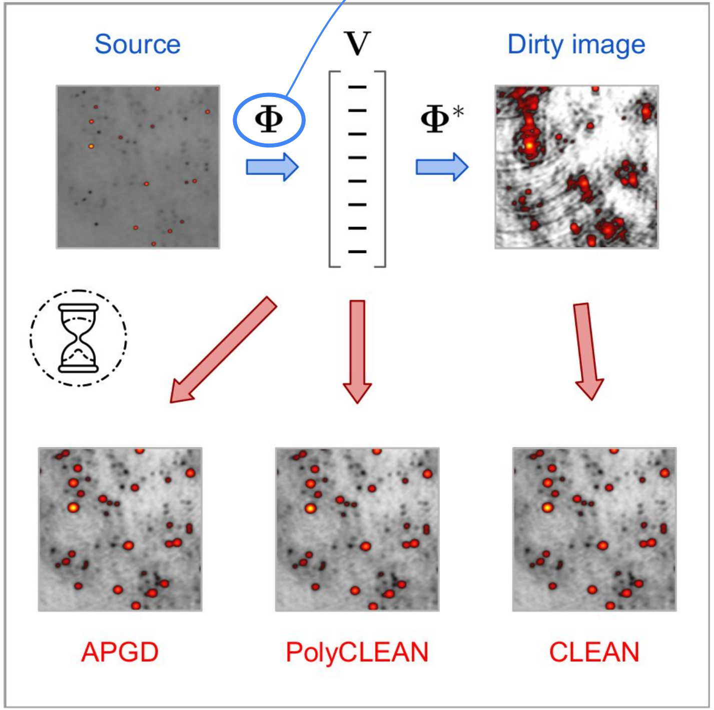
$$\mathbf{V} = \boxed{\mathbf{\Phi}}\,\mathbf{I} + \boldsymbol{\varepsilon}$$

NU Fourier sum: $V_\ell = \sum_{i,j} w_{i,j} \mathrm{e}^{-\langle \mathbf{x}_{i,j}, \mathbf{v}_\ell \rangle}$



[10] Kashani S, Queralt JR, **Jarret A**, Simeoni M. "HVOX: Scalable Interferometric Synthesis and Analysis of Spherical Sky Maps", *ArXiV pre-print*, 2023.

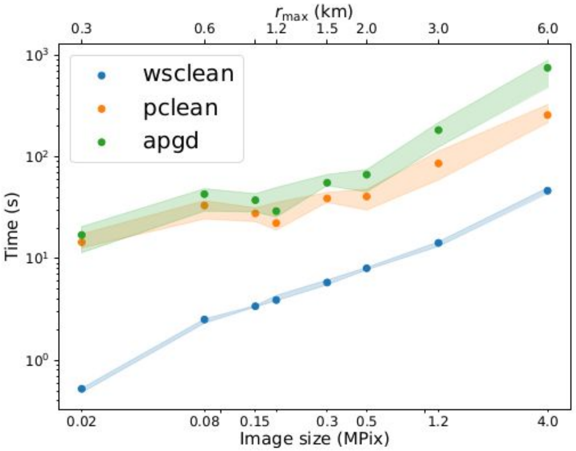
A Fast Reconstruction Method



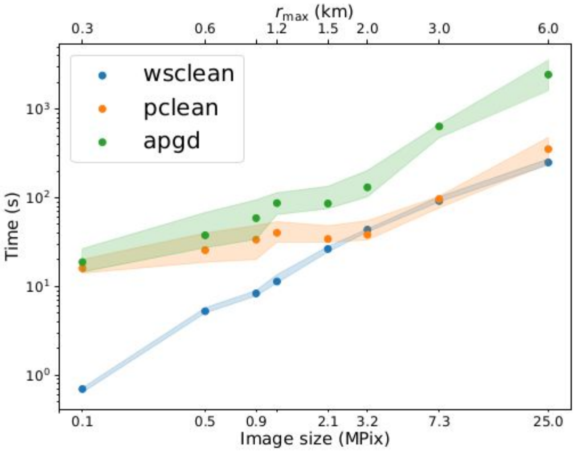
Varying number of baselines

[11] **Jarret A** et al., “PolyCLEAN: Atomic optimization for super-resolution imaging and uncertainty estimation in radio interferometry”, *Astronomy & Astrophysics*, 2025.

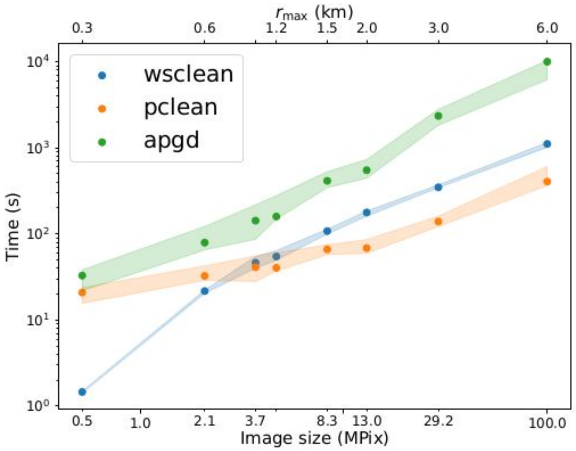
A Fast Reconstruction Method



SRF 2



SRF 5

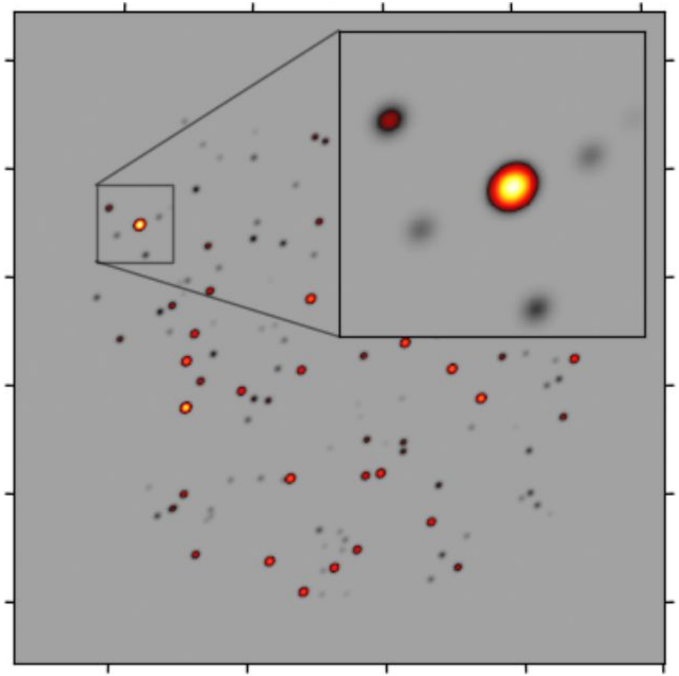


SRF 10

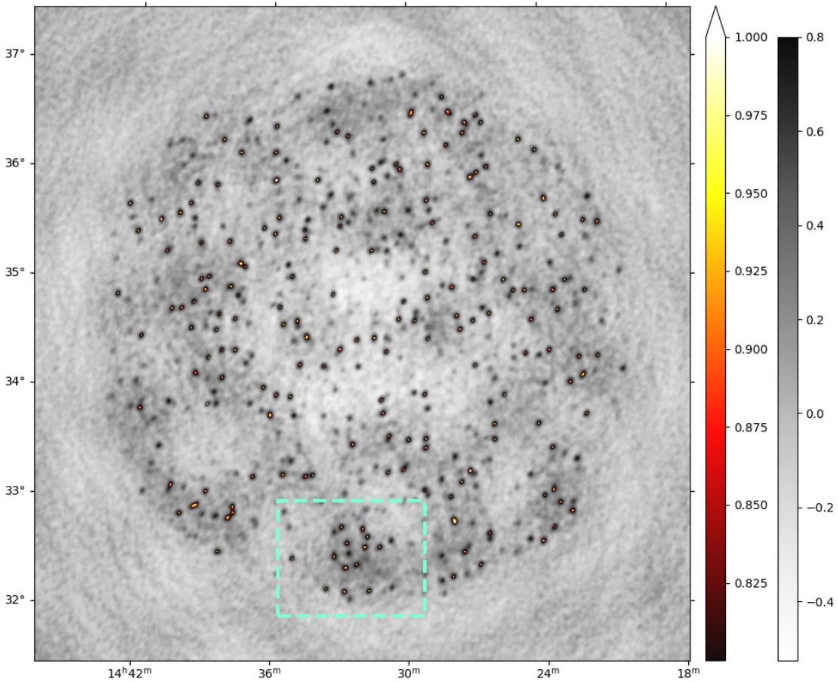
[11] **Jarret A** et al., “PolyCLEAN: Atomic optimization for super-resolution imaging and uncertainty estimation in radio interferometry”, *Astronomy & Astrophysics*, 2025.

Towards Uncertainty Estimation

$$\eta^* = \frac{1}{\lambda}\Phi^*(\mathbf{V} - \Phi\mathbf{I}^*)$$



Catalogue Boötes field

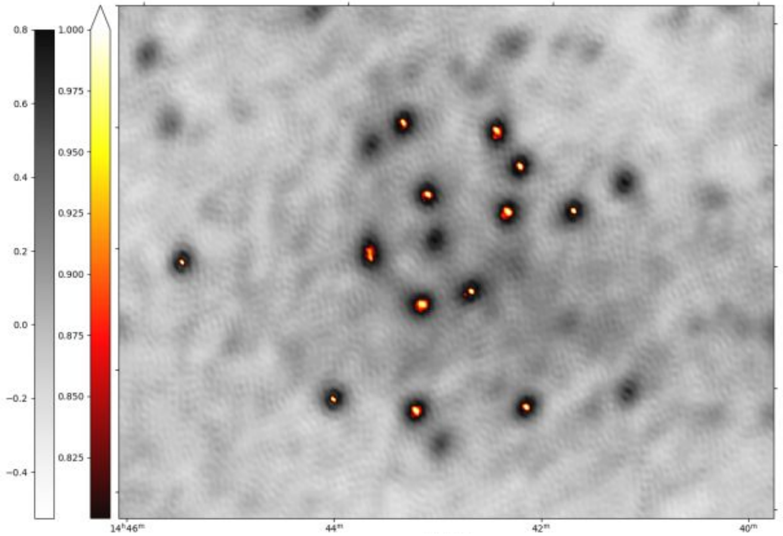


Empirical dual certificate

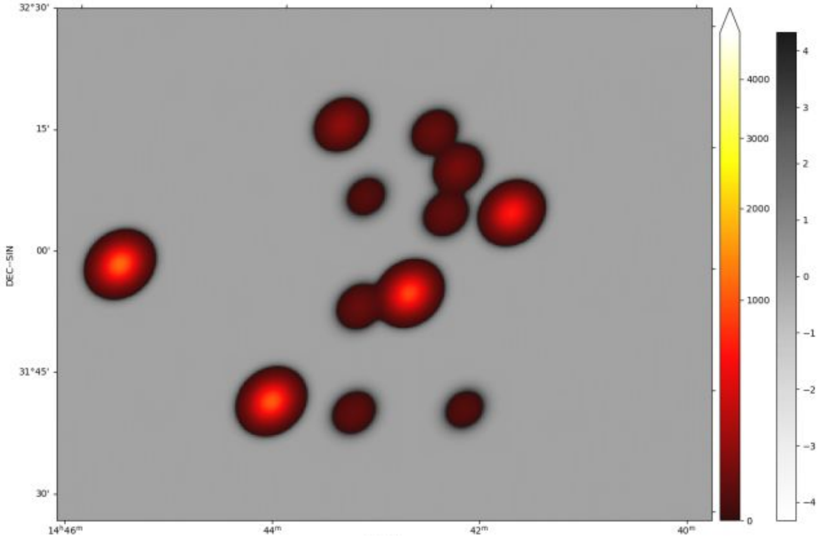
[11] **Jarret A** et al., "PolyCLEAN: Atomic optimization for super-resolution imaging and uncertainty estimation in radio interferometry", *Astronomy & Astrophysics*, 2025.

Towards Uncertainty Estimation

$$\boxed{\eta^* = \frac{1}{\lambda}\Phi^*(\mathbf{V} - \Phi\mathbf{I}^*)}$$



Dual certificate

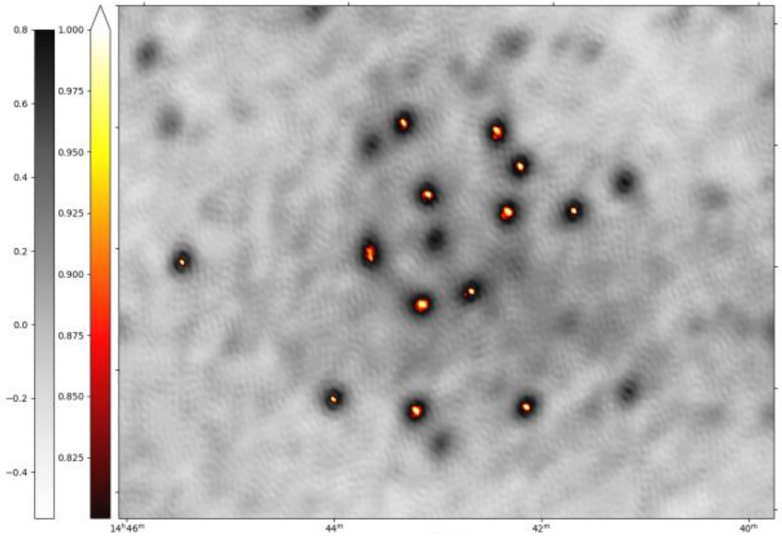


CLEAN reconstruction

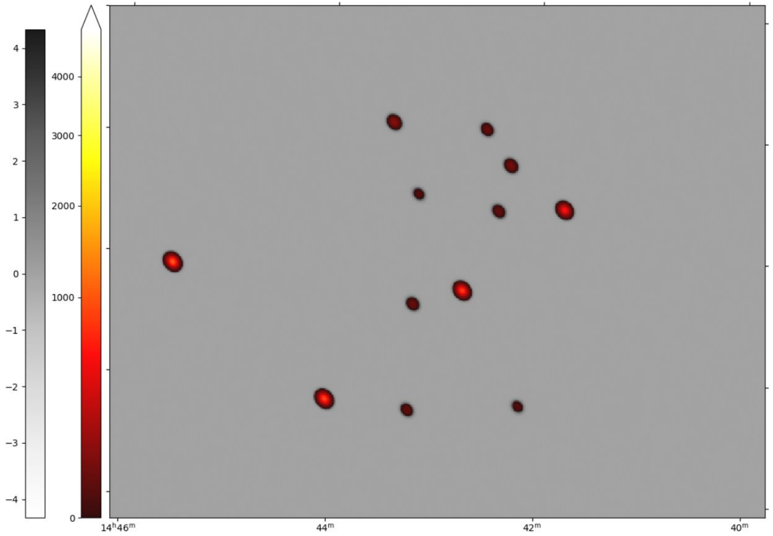
[11] **Jarret A** et al., “PolyCLEAN: Atomic optimization for super-resolution imaging and uncertainty estimation in radio interferometry”, *Astronomy & Astrophysics*, 2025.

Towards Uncertainty Estimation

$$\boxed{\color{red}{\eta^*} = \frac{1}{\lambda}\Phi^*(\mathbf{V} - \Phi\color{red}{\mathbf{I}^*})}$$



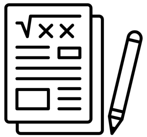
Dual certificate



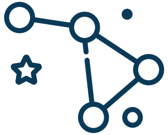
Certificate-based representation

[11] **Jarret A** et al., "PolyCLEAN: Atomic optimization for super-resolution imaging and uncertainty estimation in radio interferometry", *Astronomy & Astrophysics*, 2025.

Summary



Design of optimization algorithm → Real world application



Best of both worlds:

- Benefits of CLEAN → Atomic, fast
- Benefits of convex optimization → Accurate
- Sparsity-aware processing (HVOX) → Numerical efficiency
- Question of resolution
  → CLEAN beam is too coarse, certificate beam is data-inspired

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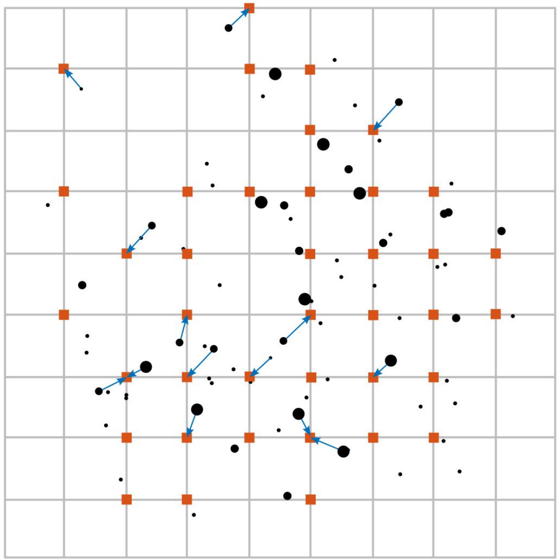
3. Conclusion

Chapter 4

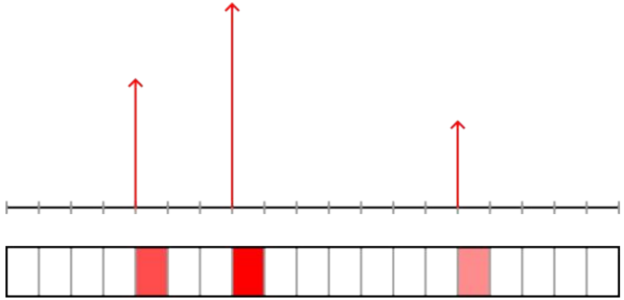


[Credits: H. Pan et al., “LEAP”, *A&A*, 2017.]

The B-LASSO Problem

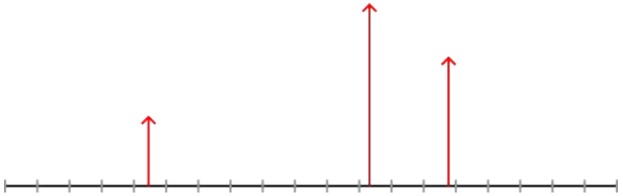
$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$$

$$\mathbf{x} =$$



$$\mathbf{y} = \boldsymbol{\Phi}(m) + \mathbf{n}$$

$$m =$$



$$\underset{m \in \mathcal{M}(\mathcal{X})}{\arg\min}\ \frac{1}{2}\|\mathbf{y} - \boldsymbol{\Phi}(m)\|_2^2 + \lambda \|m\|_{\mathcal{M}}$$ [12]

$$\|m\|_{\mathcal{M}} = \sup_{\varphi \in \mathcal{C}_0(\mathcal{X}), \|\varphi\|_\infty = 1} \langle m, \varphi \rangle$$

[12] Bredies K, Pikkarainen HK. “Inverse problems in spaces of measures”, *ESAIM: COCV,* 2013.

The B-LASSO Problem

$$\boxed{\underset{m \in \mathcal{M}(\mathcal{X})}{\arg\min} \ \frac{1}{2}\|\mathbf{y} - \mathbf{\Phi}(m)\|_2^2 + \lambda \|m\|_{\mathcal{M}}}$$ [12]

- Representer theorem[13] :

$$m^* = m[\mathbf{a}, \mathbf{x}] = \sum_i a_i \delta_{x_i}$$

$\mathbf{a} \in \mathbb{R}^K, \mathbf{x} \in \mathcal{X}^K$ $\qquad K \leq L$

- LASSO counterpart:

$$\|m[\mathbf{a}, \mathbf{x}]\|_{\mathcal{M}} = \|\mathbf{a}\|_1$$

[12] Bredies K, Pikkarainen HK. "Inverse problems in spaces of measures", *ESAIM: COCV*, 2013.

[13] Unser M. "A Unifying Representer Theorem for Inverse Problems and Machine Learning", *Foundations of Computational Mathematics*, 2020.

1. The PolyCLEAN Journey
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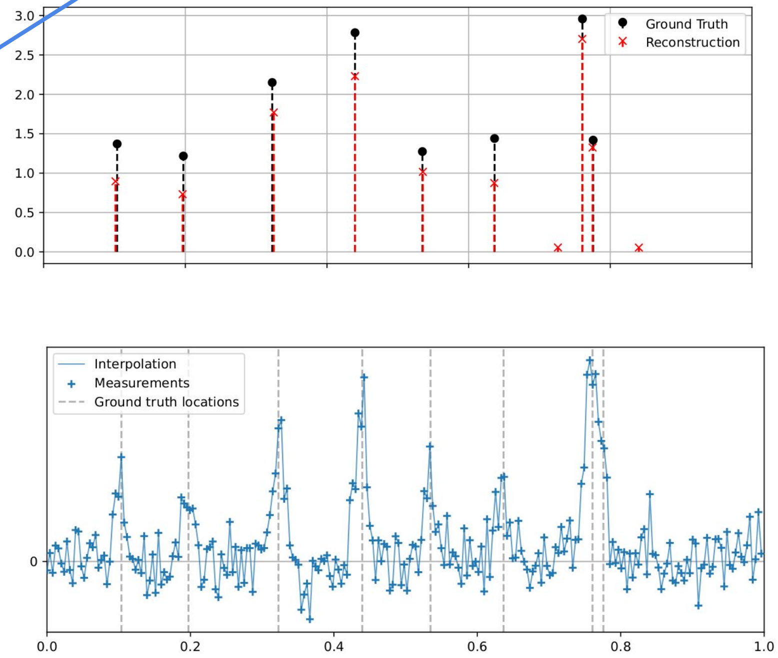
Chapter 5



Our Polyatomic Algorithm (once again)

Algorithm 5.4: Polyatomic Frank-Wolfe Algorithm for the B-LASSO

Initialize: $m_0 \leftarrow 0_{\mathcal{M}(\mathcal{X})}$, $\mathcal{S}_0 \leftarrow \emptyset$

for $k = 1, 2, \ldots$ **do**

Empirical dual certificate: $\eta_{k-1} \leftarrow \frac{1}{\lambda}\boldsymbol{\Phi}^*\left(\mathbf{y} - \boldsymbol{\Phi}(m_{k-1})\right)$

1.a) Candidate search:

$\mathcal{I}_k \leftarrow \texttt{Find\_candidates}(\eta_{k-1})$

1.b) Update active locations:

$\mathbf{x}_{k-1/2} \leftarrow \mathbf{x}_{k-1} \oplus \mathcal{I}_k$

2) Full correction of the amplitudes:

$$\mathbf{a}_{k-1/2} \in \underset{\mathbf{a}\in\mathbb{R}^{N_k}}{\arg\min}\ \frac{1}{2}\left\|\mathbf{y} - \boldsymbol{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a})\right\|_2^2 + \lambda\|\mathbf{a}\|_1$$

3) (Optional) Sliding step:

Find a local minimum of the problem

$$(\mathbf{a}_k, \mathbf{x}_k) \in \underset{(\mathbf{a},\mathbf{x})\in\mathbb{R}^{N_k}\times\mathcal{X}^{N_k}}{\arg\min}\ \frac{1}{2}\left\|\mathbf{y} - \boldsymbol{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a})\right\|_2^2 + \lambda\|\mathbf{a}\|_1$$

with initialization $(\mathbf{a}_{k-1/2}, \mathbf{x}_{k-1/2})$.

Prune the active set:

$\mathbf{x}_k \leftarrow \mathrm{Prune}(\mathbf{a}_k, \mathbf{x}_k)$

[X] **Jarret A.** et al., *Article in preparation.*

Our Polyatomic Algorithm (once again)

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Polyatomic step

Critical step !

2) Full correction of

$\mathbf{a}_{k-1/2} \in \arg$

$\mathbf{a} \in$

3) (Optional) Slidin

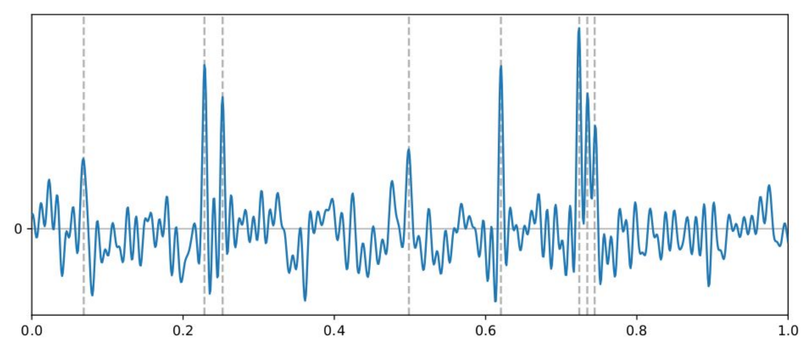
Find a local

with initialization $(\mathbf{a}_{k-1/2}, \mathbf{x}_{k-1/2})$.

Prune the active set:

$\mathbf{x}_k \leftarrow \mathrm{Prune}(\mathbf{a}_k, \mathbf{x}_k)$

Dual certificate η_0



[X] **Jarret A.** et al., *Article in preparation.*

Our Polyatomic Algorithm (once again)

Polyatomic step

Same full correction

Algorithm 5.4: Polyatomic Frank-Wolfe Algorithm for the B-LASSO

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3) (Optional) Sliding step:

Find a local minimum of the problem

$$(\mathbf{a}_k, \mathbf{x}_k) \in \underset{(\mathbf{a},\mathbf{x})\in\mathbb{R}^{N_k}\times\mathcal{X}^{N_k}}{\operatorname{argmin}} \ \frac{1}{2}\left\|\mathbf{y} - \mathbf{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a})\right\|_2^2 + \lambda\|\mathbf{a}\|_1$$

with initialization $(\mathbf{a}_{k-1/2}, \mathbf{x}_{k-1/2})$.

Prune the active set:

$\mathbf{x}_k \leftarrow \mathrm{Prune}(\mathbf{a}_k, \mathbf{x}_k)$

[X] **Jarret A.** et al., *Article in preparation.*

Our Polyatomic Algorithm (once again)

Polyatomic step

Same full correction

Optional sliding

Algorithm 5.4: Polyatomic Frank-Wolfe Algorithm for the B-LASSO

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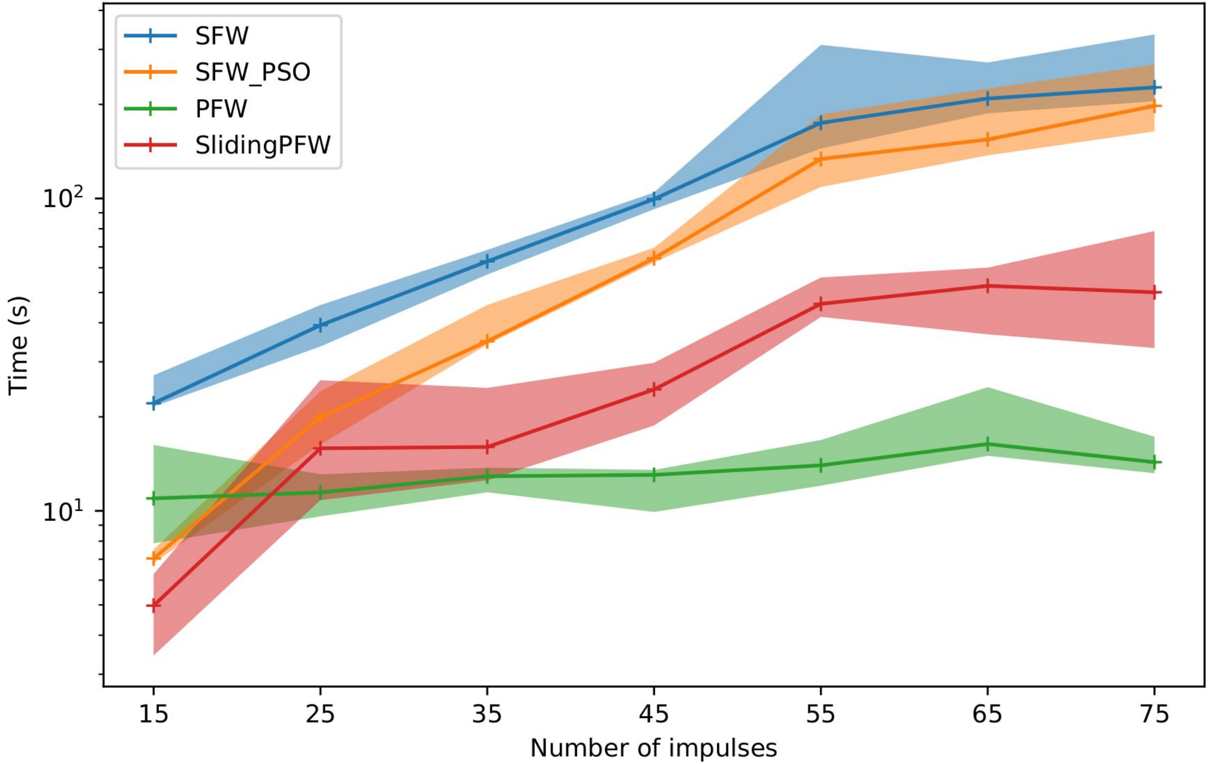
$$\mathbf{x}_k \leftarrow \mathrm{Prune}(\mathbf{a}_k, \mathbf{x}_k)$$

[X] **Jarret A.** et al., *Article in preparation.*

Promising Results



[X] **Jarret A.** et al., *Article in preparation.*

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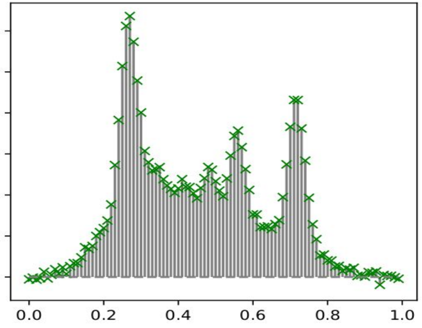
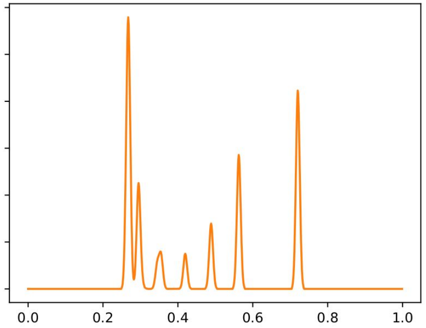
Chapter 6



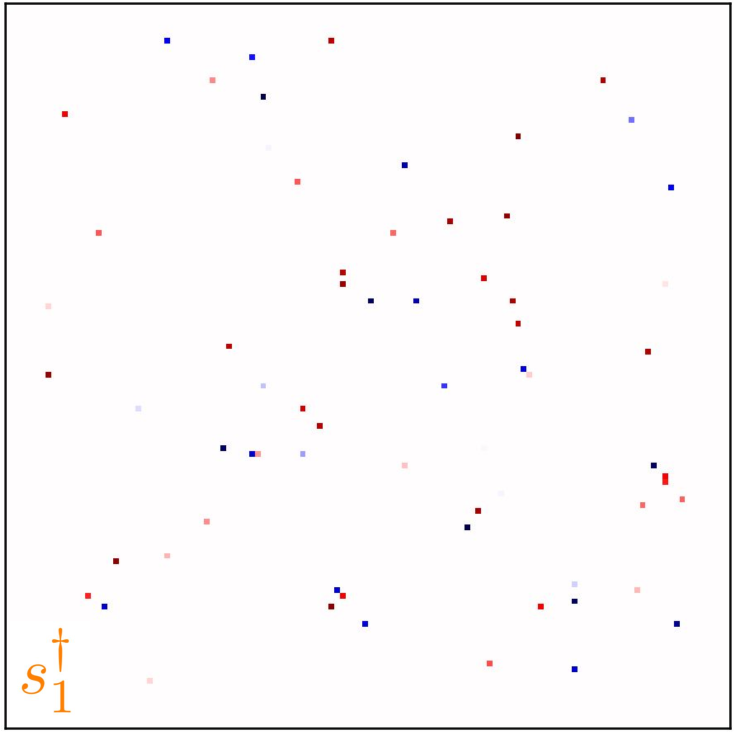


Sparse-plus-Smooth Problems

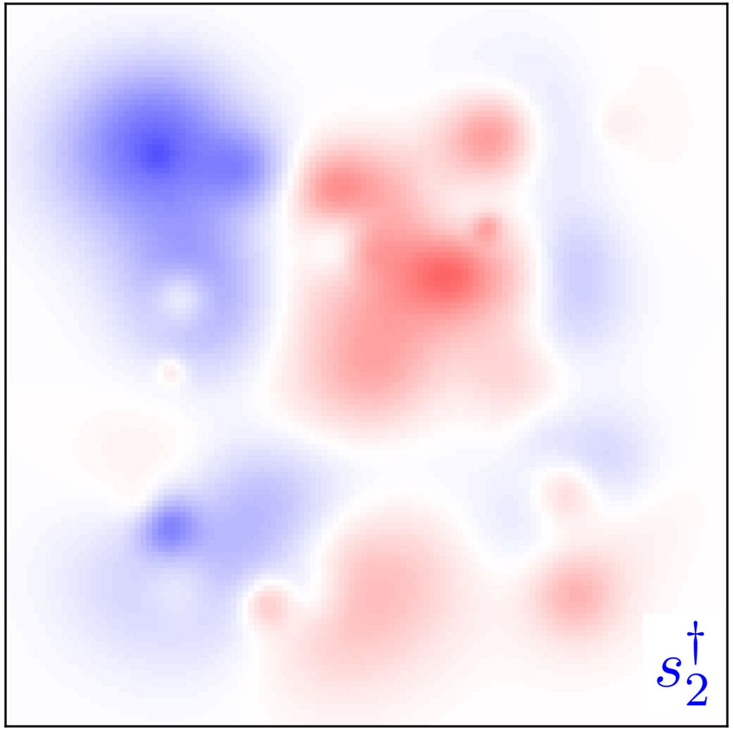
GLEAM survey of the radio sky, J2000 coordinates (9h37min15.21s, 50°25'03.1'')

Sparse-plus-Smooth Problems

$s_1^\dagger + s_2^\dagger$



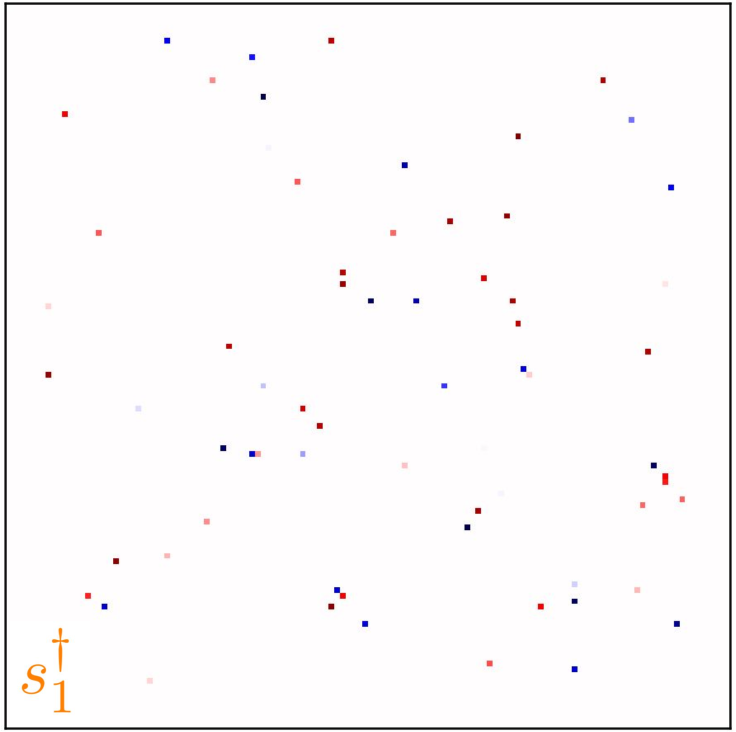
Sparse foreground



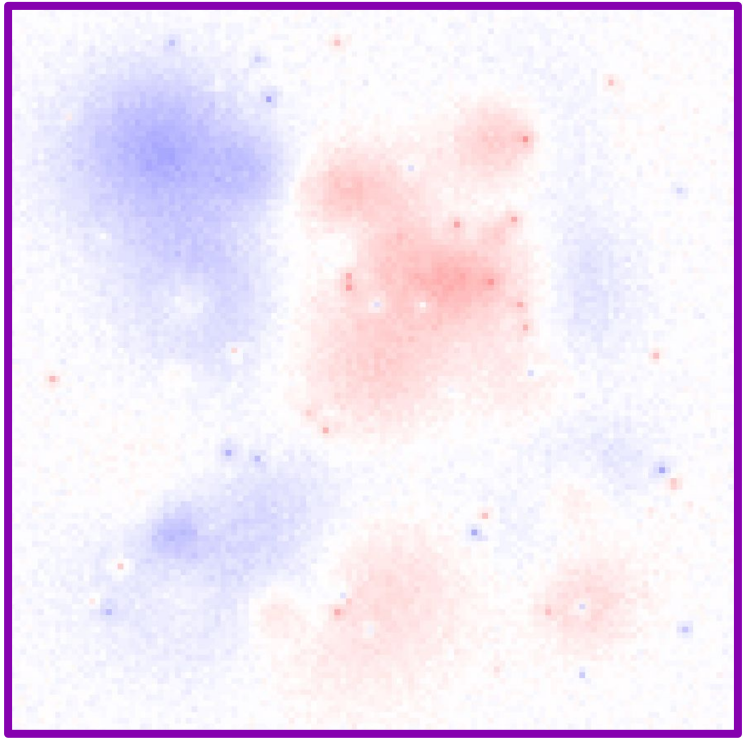
Smooth background

[8] **Jarret A** et al. "A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization", *32nd European Signal Processing Conference (EUSIPCO)*, 2024

Sparse-plus-Smooth Problems



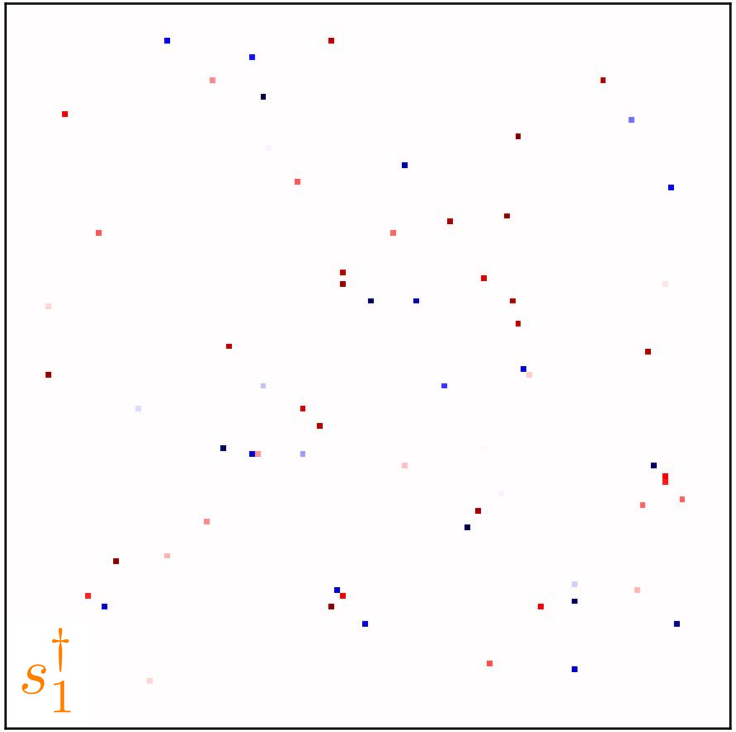
Sparse foreground



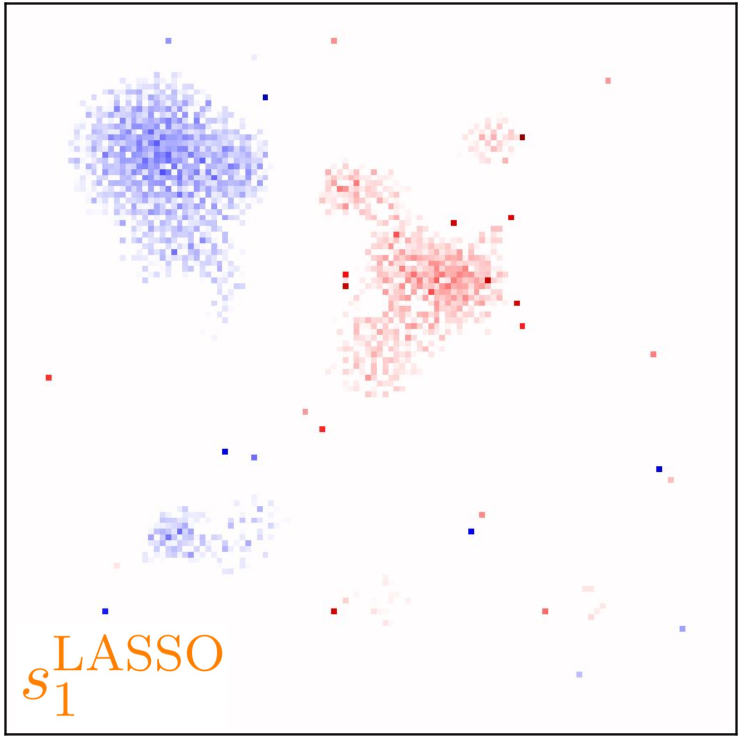
Dirty Image

[8] **Jarret A** et al. “A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization”, *32nd European Signal Processing Conference (EUSIPCO)*, 2024

Sparse-plus-Smooth Problems



Sparse foreground

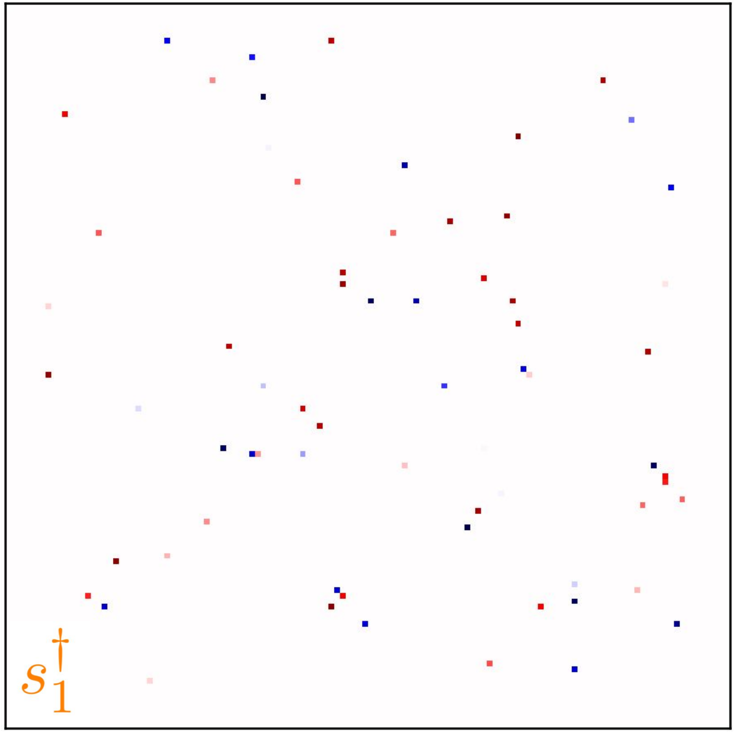


LASSO reconstruction

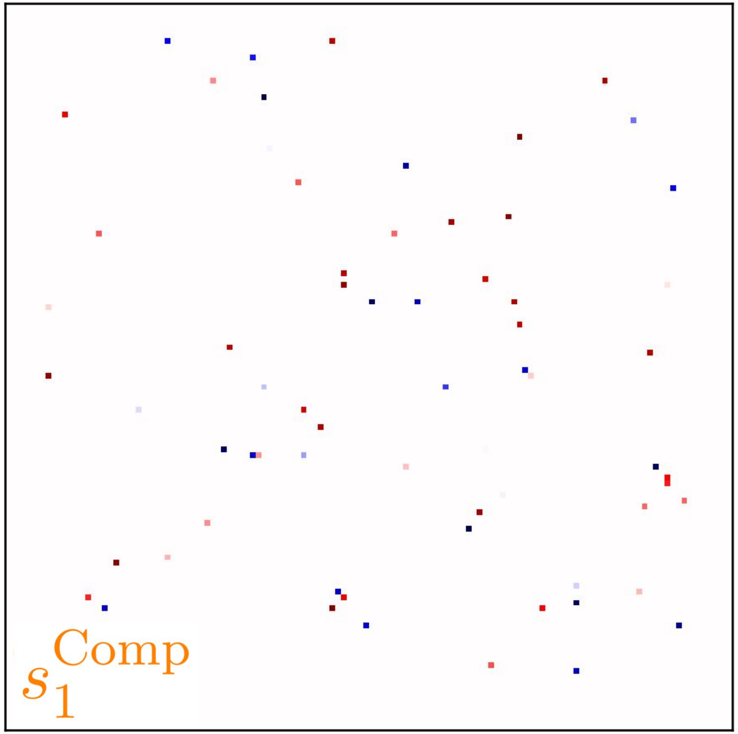
[8] **Jarret A** et al. "A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization", *32nd European Signal Processing Conference (EUSIPCO)*, 2024

Sparse-plus-Smooth Problems

$$\underset{s_1, s_2}{\arg\min}\ \mathcal{J}(s_1, s_2)$$



Sparse foreground



Composite model

[8] **Jarret A** et al. "A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization", *32nd European Signal Processing Conference (EUSIPCO)*, 2024

Composite Representer Theorem (in the literature)

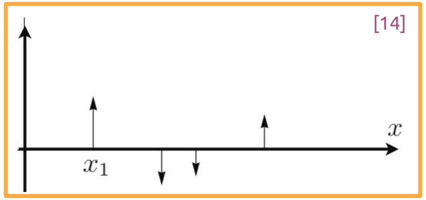
[13]

$$\underset{s_1, s_2 \in \mathcal{M}(\mathcal{X}) \times \mathrm{L}_2(\mathcal{X})}{\arg\min} \quad \frac{1}{2}\|\mathbf{y} - \mathbf{\Phi}(s_1 + s_2)\|_2^2 + \lambda_1 \|s_1\|_{\mathcal{M}} + \frac{\lambda_2}{2}\|s_2\|_{\mathrm{L}_2}^2$$

Representer theorem[13] :

- $s_1^* \rightarrow$ Sparse **measure**

$$s_1^* = m[\mathbf{a}^*, \mathbf{x}^*]$$



- $s_2^* \rightarrow$ Smooth quadratic solution

$$s_2^* = \mathbf{\Phi}^* \mathbf{u}^*$$

[13] Debarre T et al."Continuous-Domain Formulation of Inverse Problems for Composite Sparse-Plus-Smooth Signals", *IEEE Open Journal of Signal Processing*, 2021.
[14] Unser M et al., "Splines Are Universal Solutions of Linear Inverse Problems with Generalized TV Regularization", *SIAM Review*, 2017.

Our Composite Representer Theorem

$$\underset{s_1, s_2 \in \mathcal{M}(\mathcal{X}) \times \mathrm{L}_2(\mathcal{X})}{\arg\min} \; \frac{1}{2}\|\mathbf{y} - \boldsymbol{\Phi}(s_1 + s_2)\|_2^2 + \lambda_1 \|s_1\|_{\mathcal{M}} + \frac{\lambda_2}{2}\|s_2\|_{\mathrm{L}_2}^2$$

Our Representer Theorem

[Theorem 6.1]

$$\begin{cases} \widehat{s}_1 \in \underset{s_1 \in \mathcal{B}}{\arg\min} \; \frac{1}{2}\left\|\mathbf{M}_{\lambda_2}^{-\frac{1}{2}}(\mathbf{y} - \boldsymbol{\Phi}(s_1))\right\|_2^2 + \lambda_1 \|s_1\|_{\mathcal{B}} \\ \widehat{s}_2 = \frac{1}{\lambda_2}\boldsymbol{\Phi}^* \mathbf{M}_{\lambda_2}^{-1}(\mathbf{y} - \mathbf{w}) \end{cases}$$

$$\mathbf{M}_{\lambda_2} := \frac{1}{\lambda_2}(\boldsymbol{\Phi}\boldsymbol{\Phi}^* + \lambda_2 \mathbf{I}_L)$$

$$\mathbf{w} = \boldsymbol{\Phi}(\widehat{s}_1)$$

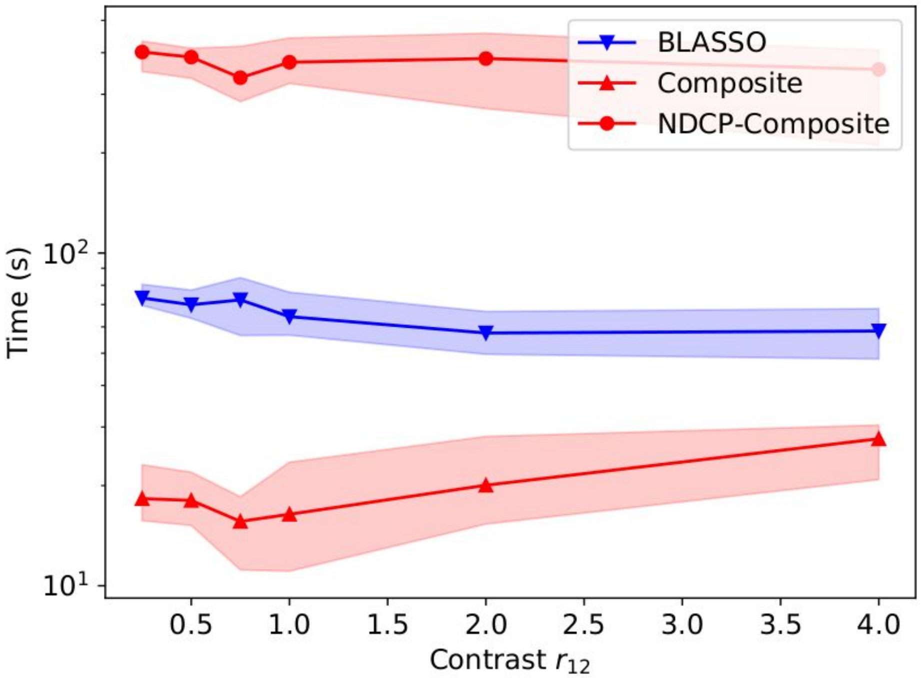
Consequences:

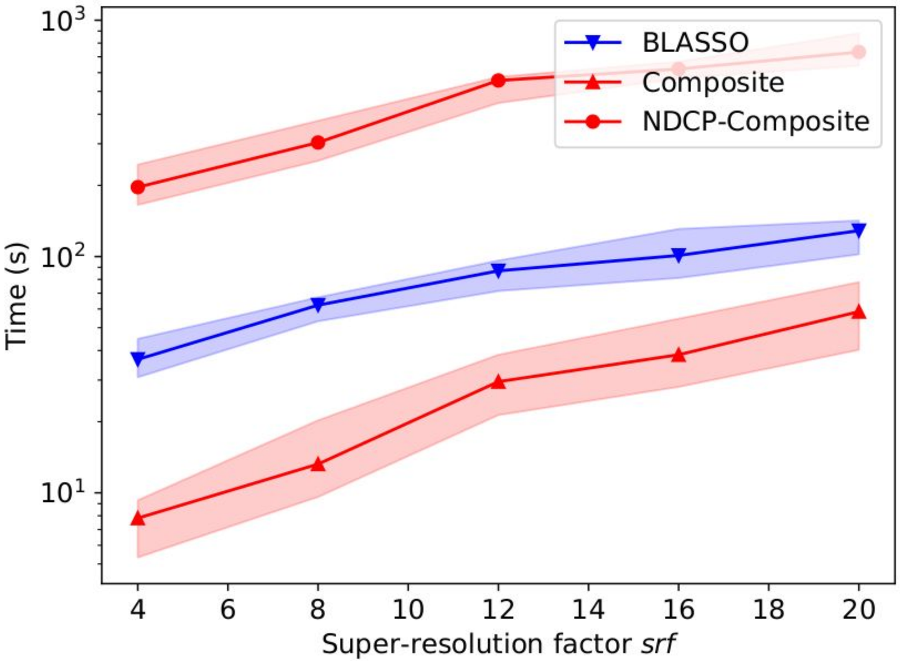
→ Decoupled reconstruction procedure

→ Scaling of regularization parameters

Advantages of a Decoupled Approach

$$r_{1/2} = \frac{\|\mathbf{\Phi}(s_1^\dagger)\|_2}{\|\mathbf{\Phi}(s_2^\dagger)\|_2}$$





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Conclusion and perspectives

Mathematics-aware numerical solvers:

- Principled (poly)atomic methods
- Decoupled algorithms
- Sparsity-aware processing

Chapters 3, 5

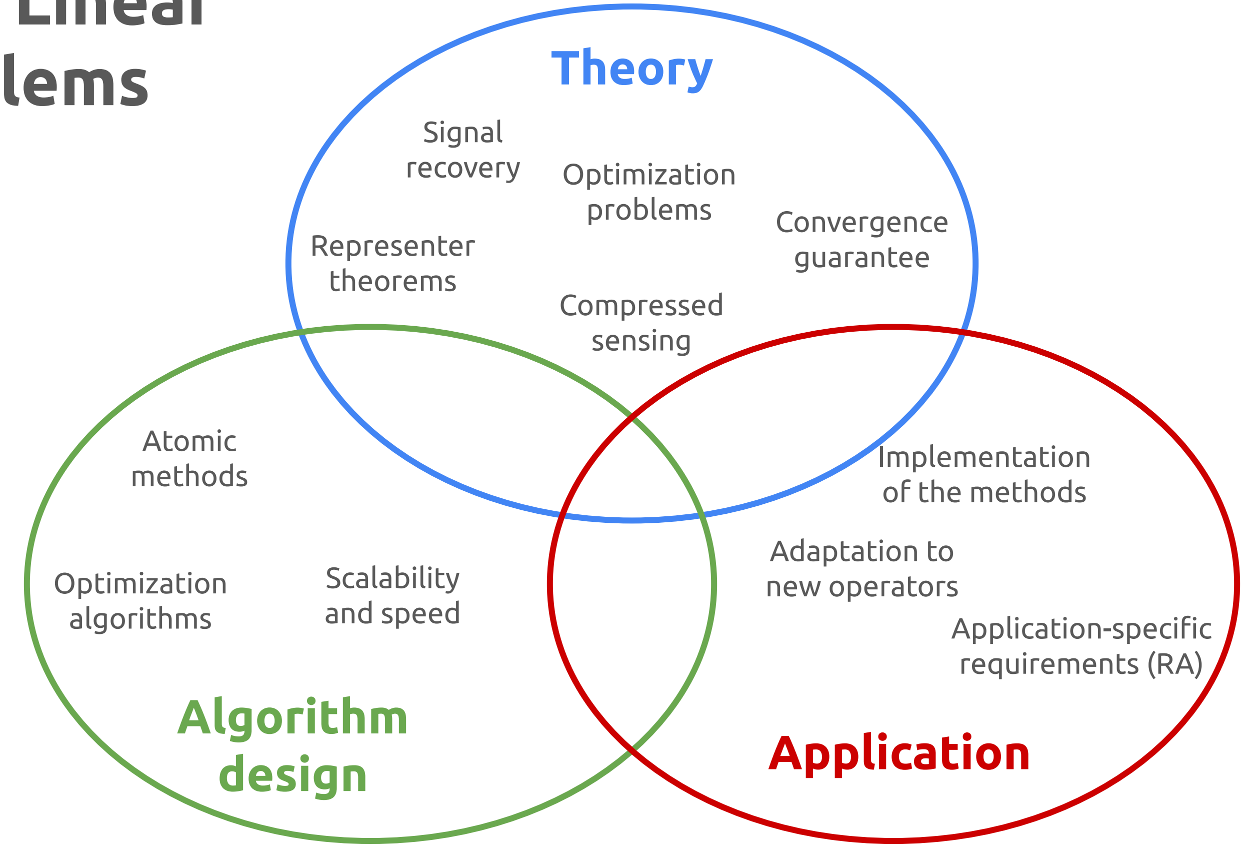
Chapter 6

Chapter 8

Open questions:

- Resolution of the reconstruction and quantitative imaging
- More advanced *traditional* methods
- Mixed learning-based approaches

Landscape of **Linear Inverse Problems**



Contributions

Part I

[Chapter 3]

Polyatomic Frank-Wolfe for the LASSO

Jarret A, Fageot J, Simeoni M,
"A Fast and Scalable Polyatomic Frank-Wolfe Algorithm for the LASSO",
IEEE Signal Processing Letters, 2022.

+ GRETSI 2022

Part II

[Chapter 5]

Polyatomic Continuous-Domain Reconstruction

Jarret A, Rochinha-Chaves D, Denoyelle Q, Vetterli M,
Article in preparation.

Part II

[Chapter 6]

Decoupling of Composite Sparse-plus-Smooth problems

Jarret A, Fageot J,
"Decoupled Solution for Composite Sparse-plus-Smooth Inverse Problems",
Submitted in June 2025.

Jarret A, Costa V, Fageot J,
"A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization",
Proceeding of EUSIPCO 2024.

Part III

[Chapter 8]

Radio Interferometric Imaging with PolyCLEAN

Jarret A, Kashani S, Rué-Queralt J, Hurley P, Fageot J, Simeoni M,
"PolyCLEAN: Atomic optimization for super-resolution imaging and uncertainty estimation in radio interferometry",
Astronomy and Astrophysics, 2025

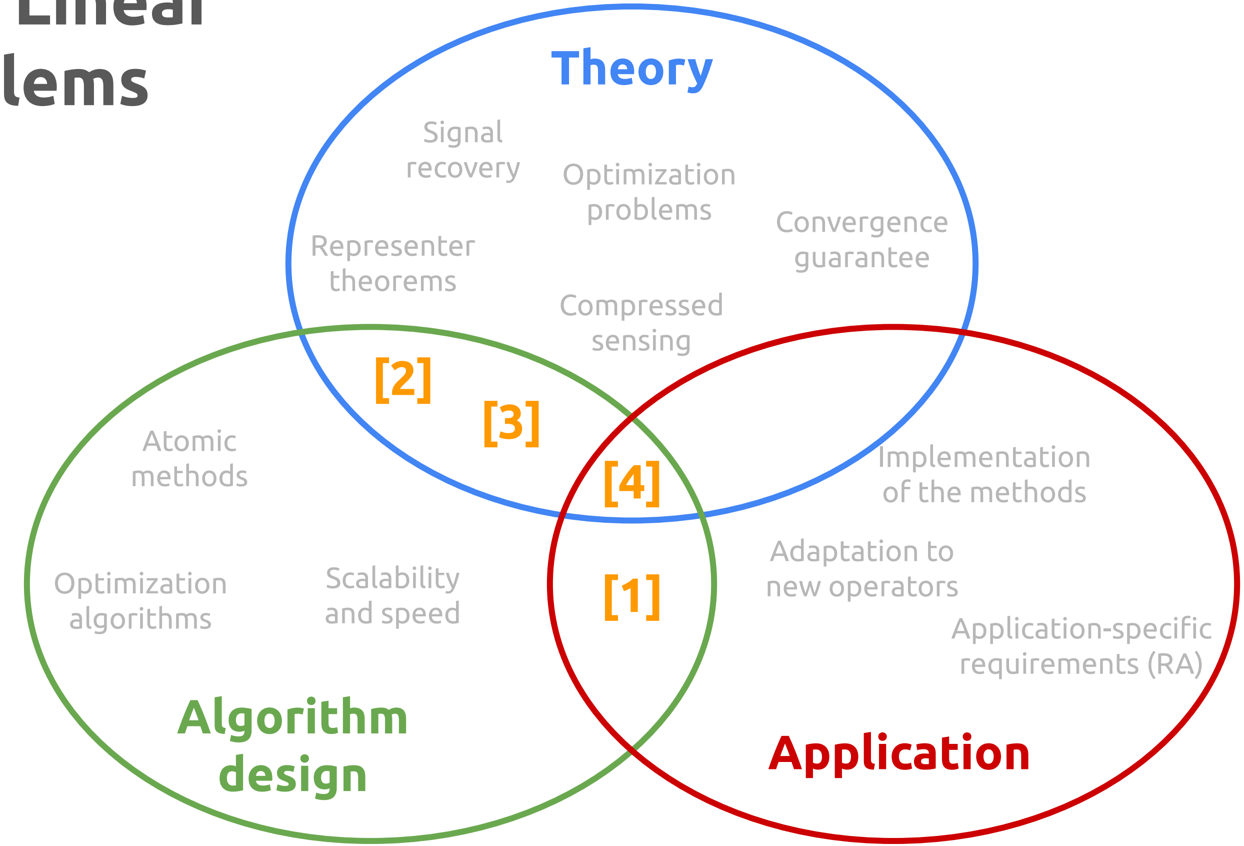
Landscape of **Linear Inverse Problems**

[1] PFW

[2] CD-PFW

[3] Composite

[4] PolyCLEAN



Thank you !

Advisors and collaborators:

Martin Vetterli
Julien Fageot
Matthieu Simeoni
Paul Hurley

Sepand Kashani
Quentin Denoyelle
David Rochinha-Chaves
Valérie Costa

Labmates (past and present):

LCAV - IVRL - Center for Imaging

Supplementary slides

Penalized Optimization

Discrete problems

$$\underset{\mathbf{x}\in\mathbb{R}^N}{\arg\min}\ \underbrace{\frac{1}{2}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2}_{\text{Data-fidelity}} + \underbrace{\mathcal{R}(\mathbf{x})}_{\text{Penalty}}$$

LASSO[1]

$$\mathcal{R}(\mathbf{x}) = \lambda\|\mathbf{x}\|_1$$

Continuous-domain problems

$$\underset{f\in\mathcal{M}(\mathbb{R}^d)}{\arg\min}\ \underbrace{\frac{1}{2}\|\mathbf{y}-\boldsymbol{\Phi}(f)\|_2^2}_{\text{Data-fidelity}} + \underbrace{\mathcal{R}(f)}_{\text{Penalty}}$$

B-LASSO[2]

$$\mathcal{R}(f) = \lambda\|f\|_{\mathcal{M}}$$

[1] Tibshirani R. “Regression Shrinkage and Selection via the Lasso”, *Journal of the Royal Statistical Society Series B (Methodological)*, 1996.
[2] Bredies K, Pikkarainen HK. “Inverse problems in spaces of measures”, *ESAIM: COCV*, 2013.

2. Regularized Optimization *(continued)*

Representer theorem: [3]

$$\mathbf{x}^* = \sum_{i=1}^{L} a_i \mathbf{e}_i$$

$$f^* = \sum_{i=1}^{L} a_i \delta_{x_i}$$

Benefits of the optimization approach:

- Implicit model
- Decorrelate methodology and implementation
- Versatility

- Understandability (objective function)
- Principled: exact reconstruction in low noise regime
- Bayesian interpretation

[3] Unser M. "A Unifying Representer Theorem for Inverse Problems and Machine Learning", *Foundations of Computational Mathematics*, 2020.

1. **The PolyCLEAN Journey**
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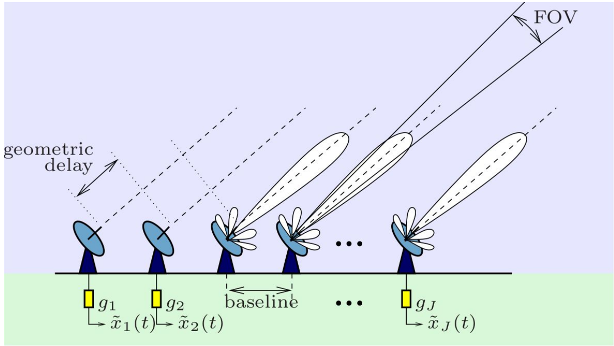
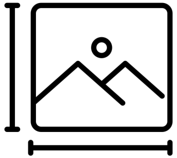
Inverse problem in radio astronomy



- Noisy measurements $\mathbf{V} = \mathbf{\Phi} \mathbf{I} + \varepsilon$
- Ill-posed problem $\mathrm{Null}(\mathbf{\Phi}) \neq \{0\}$
- Huge volumes of data



[8] Van der Veen et al. "Signal Processing for Radio Astronomy", 2019

Conventional solving methods

- The CLEAN family :

✅ Simple and accelerated -> fast
✅ Many variants:
  - MS-CLEAN, MFS-CLEAN, …
✅ Long term standard
✅ Calibration

❌ Sensitive to stop
❌ No denoising
❌ Objective function unclear
❌ Physically impossible artefacts

- The optimization methods :

✅ Principled and controlled solutions
✅ Optimization solvers
✅ Versatile priors
✅ Uncertainty quantification
✅ Active field
✅ Improved results

❌ Potentially slow to converge
❌ Numerically heavy
❌ Little adoption in the field

The PolyCLEAN framework

PolyCLEAN

1. LASSO
2. Polyatomic FW
3. HVOX forward operator

Benefits:

- Fast and scalable
- Accuracy of optimization methods
- Dual certificate image

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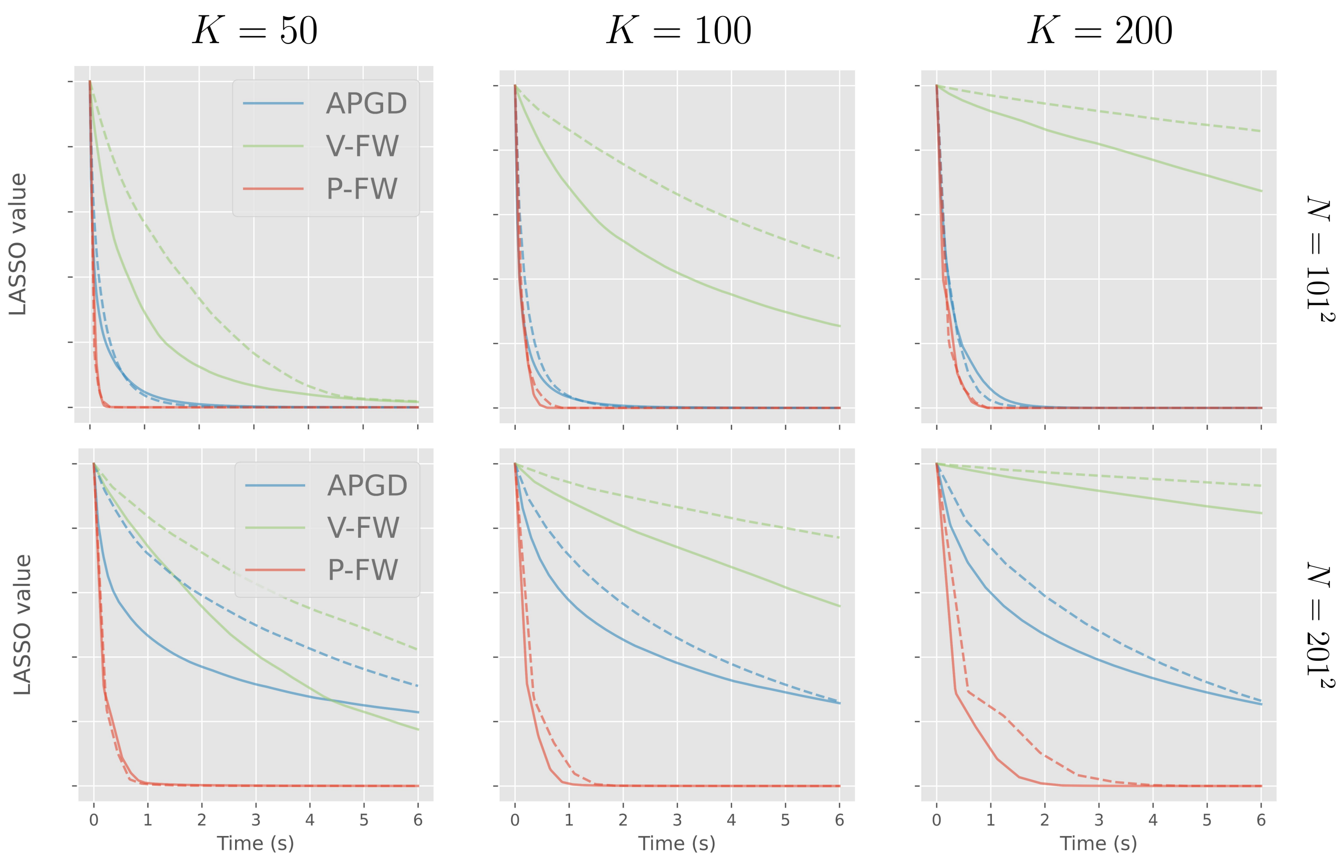
Chapter 3

Simulations

- Simulated LASSO problem:

  $$\mathbf{A} \in \mathbb{R}^{L \times K}$$
  $$\mathbf{y} \in \mathbb{R}^{L}$$
  $$L = 8K \text{ or } 16K$$

- Benefits:
  - Faster
  - Dependency on K



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Chapter 8

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Chapter 4

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Chapter 5

The B-LASSO for Sparse Continuous-Domain Recovery

$$\arg\min_{m \in \mathcal{M}(\mathcal{X})} \frac{1}{2}\|\mathbf{y} - \mathbf{\Phi}(m)\|_2^2 + \lambda\|m\|_{\mathcal{M}}$$

$$\mathcal{X} = \mathbb{R}^d$$

$$\mathcal{X} = \mathbb{T}^d$$

$$\mathcal{A} = (\mathcal{C}_0(\mathcal{X}), \|\cdot\|_\infty)$$

$$\mathcal{M}(\mathcal{X}) = \mathcal{B} = \mathcal{A}'$$

$$\|m\|_{\mathcal{M}} = \sup_{\varphi \in \mathcal{C}_0(\mathcal{X}), \|\varphi\|_\infty = 1} \langle m, \varphi \rangle = \|m\|_*$$

$$m[\mathbf{a}, \mathbf{x}] = \sum_i a_i \delta_{x_i}, \quad \mathbf{a} \in \mathbb{R}^K, \mathbf{x} \in \mathcal{X}^K, \quad \|m[\mathbf{a}, \mathbf{x}]\|_{\mathcal{M}} = \|\mathbf{a}\|_1$$

[3] Unser M. "A Unifying Representer Theorem for Inverse Problems and Machine Learning", *Foundations of Computational Mathematics*, 2020.

The Sliding Frank-Wolfe Algorithm

Algorithm 5.3: Sliding Frank-Wolfe for the B-LASSO

$\boxed{m_k = m[\mathbf{a}_k, \mathbf{x}_k]}$

Initialize: $\mathbf{a}_0 \leftarrow [\,]$, $\mathbf{x}_0 \leftarrow [\,]$

for $k = 1, 2, \ldots$ **do**

Empirical dual certificate: $\eta_{k-1} \leftarrow \frac{1}{\lambda}\boldsymbol{\Phi}^* \left(\mathbf{y} - \boldsymbol{\Phi}(m_{k-1})\right)$

1.a) New impulse location:

$x_k \in \arg\max_{x \in \mathcal{X}} |\eta_{k-1}(x)|$

1.b) Update active locations:

$\mathbf{x}_{k-1/2} \leftarrow \mathbf{x}_{k-1} \oplus x_k$ `// Concatenation`

2) Full correction of the amplitudes:

$$\mathbf{a}_{k-1/2} \in \underset{\mathbf{a} \in \mathbb{R}^{N_k}}{\arg\min}\ \frac{1}{2}\left\|\mathbf{y} - \boldsymbol{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a})\right\|_2^2 + \lambda\|\mathbf{a}\|_1$$

3) Sliding step:

Find a local minimum of the problem

$$(\mathbf{a}_k, \mathbf{x}_k) \in \underset{(\mathbf{a},\mathbf{x}) \in \mathbb{R}^{N_k} \times \mathcal{X}^{N_k}}{\arg\min}\ \frac{1}{2}\left\|\mathbf{y} - \boldsymbol{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a})\right\|_2^2 + \lambda\|\mathbf{a}\|_1$$

with initialization $(\mathbf{a}_{k-1/2}, \mathbf{x}_{k-1/2})$.

Prune the active set:

$\mathbf{x}_k \leftarrow \mathrm{Prune}(\mathbf{a}_k, \mathbf{x}_k)$

[5] Denoyelle Q et al. “The sliding Frank–Wolfe algorithm and its application to super-resolution microscopy”, *Inverse Problems*, 2019.

The Sliding Frank-Wolfe Algorithm

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[5] Denoyelle Q et al. "The sliding Frank–Wolfe algorithm and its application to super-resolution microscopy", *Inverse Problems*, 2019.

The Sliding Frank-Wolfe Algorithm

"Fully-Corrective Continuous-Domain Frank-Wolfe algorithm"

Convergence:

$$P(m_k) - P^* = \mathcal{O}(1/k)$$

Algorithm 5.3: Sliding Frank-Wolfe for the B-LASSO

$$\boxed{m_k = m[\mathbf{a}_k, \mathbf{x}_k]}$$

Initialize: $\mathbf{a}_0 \leftarrow [\,]$, $\mathbf{x}_0 \leftarrow [\,]$

for $k = 1, 2, \dots$ **do**

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$$\mathbf{x}_{k-1/2} \leftarrow \mathbf{x}_{k-1} \oplus x_k \quad \text{// Concatenation}$$

2) Full correction of the amplitudes:

$$\mathbf{a}_{k-1/2} \in \operatorname*{argmin}_{\mathbf{a} \in \mathbb{R}^{N_k}} \frac{1}{2}\left\|\mathbf{y} - \boldsymbol{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a})\right\|_2^2 + \lambda \|\mathbf{a}\|_1$$

3) Sliding step:

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The Sliding Frank-Wolfe Algorithm

- ✅ Finite steps exact convergence (under mild assumptions)
- ❌ Computationally heavy steps (candidate search and sliding)

Algorithm 5.3: Sliding Frank-Wolfe for the B-LASSO

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$$\mathbf{x}_{k-1/2} \leftarrow \mathbf{x}_{k-1} \oplus x_k$$

// Concatenation

2) Full correction of the amplitudes:

$$\mathbf{a}_{k-1/2} \in \arg\min_{\mathbf{a} \in \mathbb{R}^{N_k}} \frac{1}{2} \left\| \mathbf{y} - \mathbf{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a}) \right\|_2^2 + \lambda \|\mathbf{a}\|_1$$

3) Sliding step:

Find a local minimum of the problem

$$(\mathbf{a}_k, \mathbf{x}_k) \in \arg\min_{(\mathbf{a},\mathbf{x}) \in \mathbb{R}^{N_k} \times \mathcal{X}^{N_k}} \frac{1}{2} \left\| \mathbf{y} - \mathbf{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a}) \right\|_2^2 + \lambda \|\mathbf{a}\|_1$$

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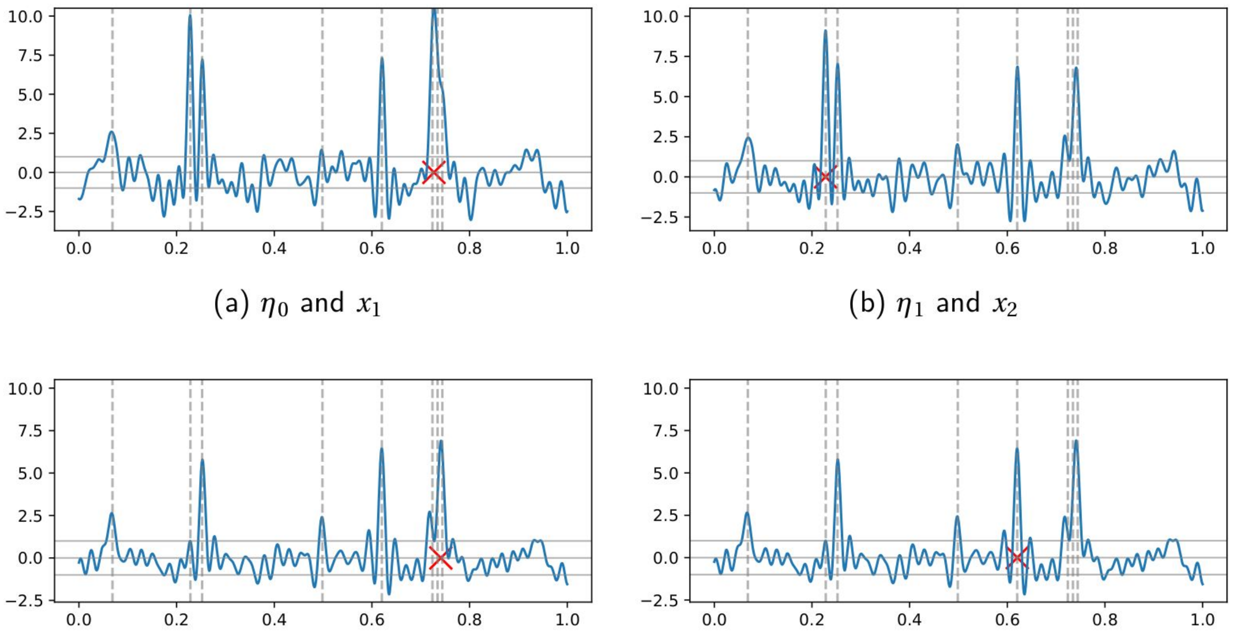
The Sliding Frank-Wolfe Algorithm



(a) η_0 and x_1 (b) η_1 and x_2

(c) η_2 and x_3 (d) η_3 and x_4

Our Polyatomic Algorithm (once again)

Algorithm 5.4: Polyatomic Frank-Wolfe Algorithm for the B-LASSO

Initialize: $m_0 \leftarrow 0_{\mathcal{M}(\mathcal{X})}$, $\mathcal{S}_0 \leftarrow \emptyset$

for $k = 1, 2, \dots$ **do**

Empirical dual certificate: $\eta_{k-1} \leftarrow \frac{1}{\lambda}\mathbf{\Phi}^*\left(\mathbf{y} - \mathbf{\Phi}(m_{k-1})\right)$

1.a) Candidate search:

$\mathcal{I}_k \leftarrow \texttt{Find\_candidates}(\eta_{k-1})$

1.b) Update active locations:

$\mathbf{x}_{k-1/2} \leftarrow \mathbf{x}_{k-1} \oplus \mathcal{I}_k$ // Concatenation

2) Full correction of the amplitudes:

$$\mathbf{a}_{k-1/2} \in \operatorname*{argmin}_{\mathbf{a} \in \mathbb{R}^{N_k}} \frac{1}{2}\left\|\mathbf{y} - \mathbf{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a})\right\|_2^2 + \lambda\|\mathbf{a}\|_1$$

3) (Optional) Sliding step:

Find a local minimum of the problem

$$(\mathbf{a}_k, \mathbf{x}_k) \in \operatorname*{argmin}_{(\mathbf{a},\mathbf{x}) \in \mathbb{R}^{N_k} \times \mathcal{X}^{N_k}} \frac{1}{2}\left\|\mathbf{y} - \mathbf{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a})\right\|_2^2 + \lambda\|\mathbf{a}\|_1$$

with initialization $(\mathbf{a}_{k-1/2}, \mathbf{x}_{k-1/2})$.

Prune the active set:

$\mathbf{x}_k \leftarrow \mathrm{Prune}(\mathbf{a}_k, \mathbf{x}_k)$

Our Polyatomic Algorithm (once again)

Polyatomic step

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Our Polyatomic Algorithm (once again)

Polyatomic step

Same full correction

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3) (Optional) Sliding step:

Find a local minimum of the problem

$$(\mathbf{a}_k, \mathbf{x}_k) \in \underset{(\mathbf{a},\mathbf{x}) \in \mathbb{R}^{N_k} \times \mathcal{X}^{N_k}}{\arg\min}\ \frac{1}{2}\left\|\mathbf{y} - \mathbf{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a})\right\|_2^2 + \lambda \|\mathbf{a}\|_1$$

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Prune the active set:

$\mathbf{x}_k \leftarrow \mathrm{Prune}(\mathbf{a}_k, \mathbf{x}_k)$

Our Polyatomic Algorithm (once again)

Polyatomic step

Same full correction

Optional sliding

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3) (Optional) Sliding step:

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$$(\mathbf{a}_k, \mathbf{x}_k) \in \underset{(\mathbf{a},\mathbf{x})\in\mathbb{R}^{N_k}\times\mathcal{X}^{N_k}}{\operatorname{argmin}} \ \frac{1}{2}\left\|\mathbf{y} - \mathbf{\Phi}_{\mathbf{x}_{k-1/2}}(\mathbf{a})\right\|_2^2 + \lambda \|\mathbf{a}\|_1$$

with initialization $(\mathbf{a}_{k-1/2}, \mathbf{x}_{k-1/2})$.

Prune the active set:

$\mathbf{x}_k \leftarrow \mathrm{Prune}(\mathbf{a}_k, \mathbf{x}_k)$

Choice of the atoms

1.a) Candidate search:
$\mathcal{I}_k \leftarrow \texttt{Find\_candidates}(\eta_{k-1})$

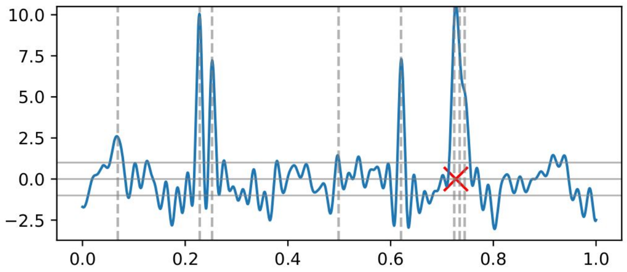
Critical step:

- Global optimality
- Relevant candidates (make progress)
- Spatial diversity

$$\left(\underset{x\in\mathcal{X}}{\arg\max}\ |\eta_{k-1}(x)|\right) \cap \mathcal{I}_k \neq \emptyset$$

$$\forall x \in \mathcal{I}_k, \quad |\eta_{k-1}(x)| \geq 1$$

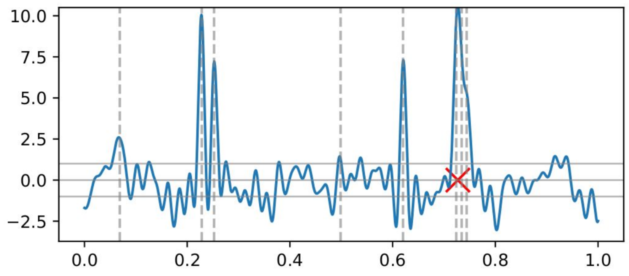
Target: Local maxima of the dual certificate



(a) η_0 and x_1

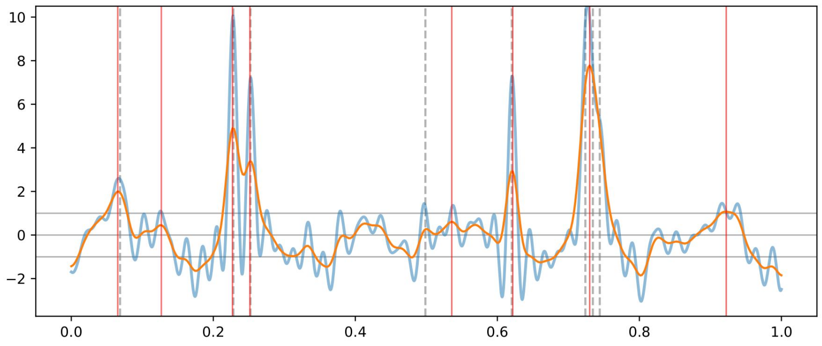
Candidate Selection Strategies

- Particles swarm optimization:
  - Fast, 0^{th}-order
  - Initialization-dependent, lack of accuracy and stability

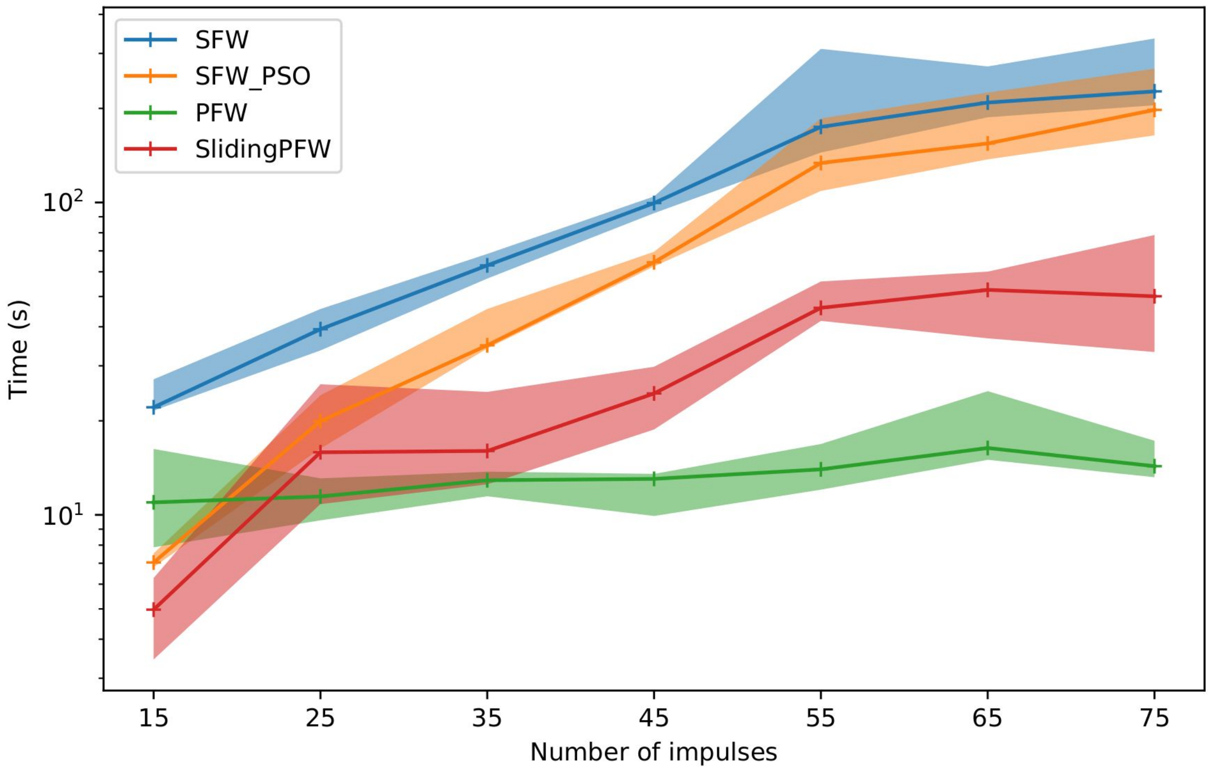


- Particles gradient descent (p-GD)
  - Locally optimal
  - Initialization-dependent, computationally heavy

- Smoothing initialization
  - Filtering of non-relevant candidates



Results in Simulations



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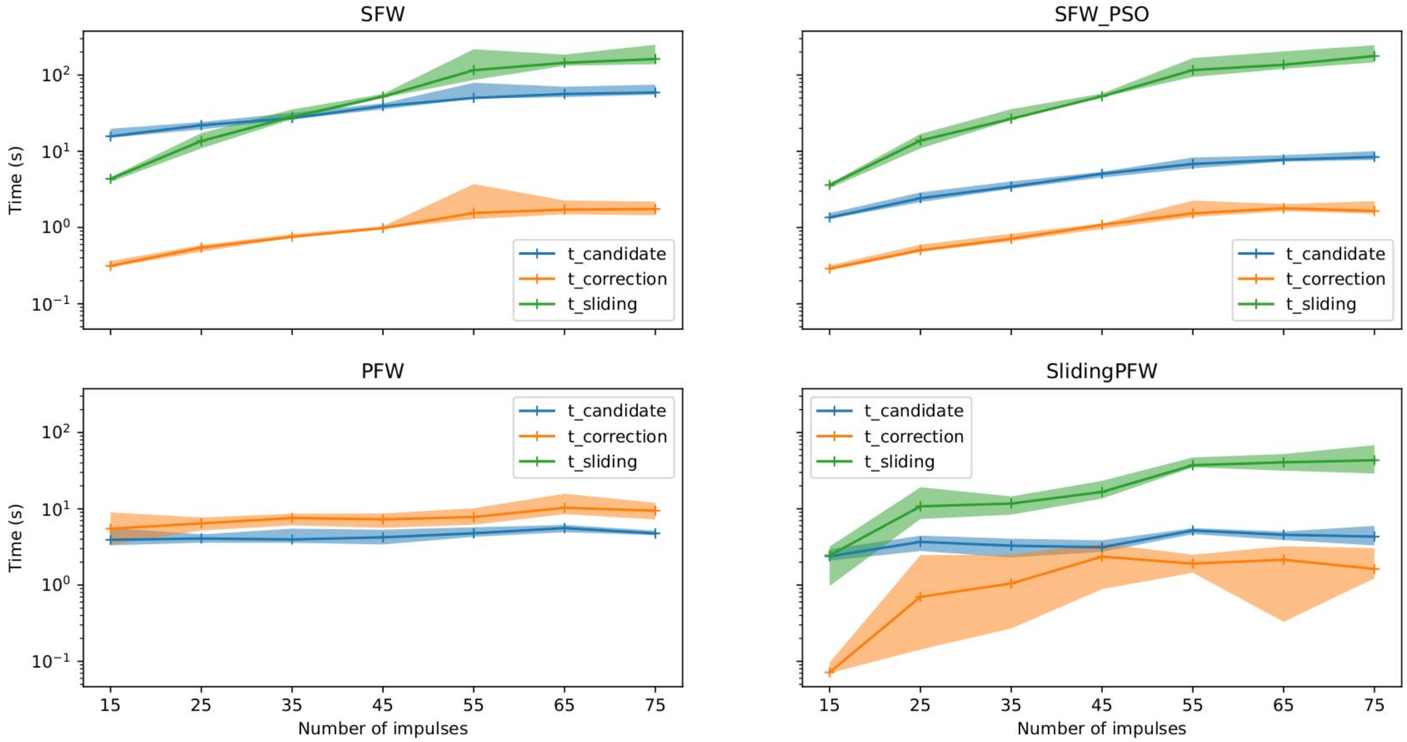


Conclusions

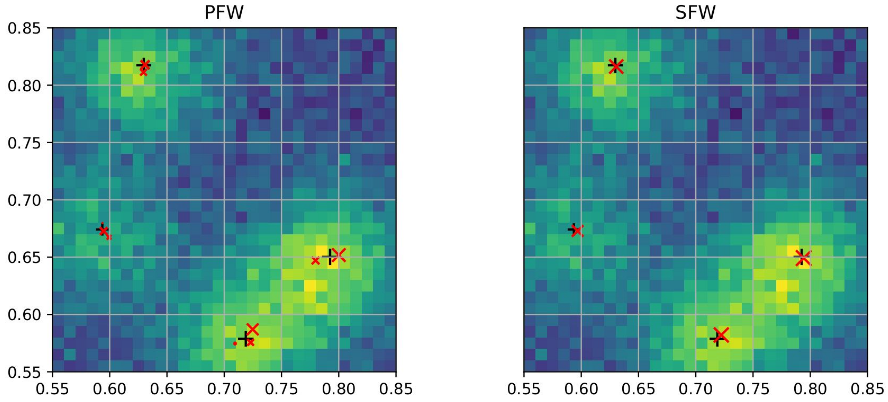
- PFW is faster in challenging contexts (high dimensions, large number of Dirac impulses)
- May be less precise than the reconstruction of SFW

Alternatives and solutions (WIP):

- Mix a few sliding steps (akin to minor/major cycles of CLEAN)
- Sparsify the solution (simplex algorithm)

	PFW	SFW
Recovery time (s)	13.8	24.9
Time for:		
- Candidate search	9.7	10.1
- Full correction	4.2	0.2
- Sliding	—	14.5
Objective function ($\times 10^6$)	4.373	4.370
Flat metric:		
- parameter $= 0.002$	0.031	0.027
- parameter $= 0.01$	0.085	0.058

Results for 2D Gaussian measurements



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Chapter 6

Composite Sparse-plus-Smooth Problems

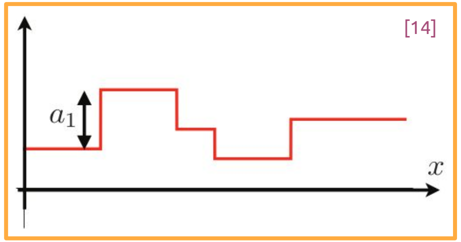
Sparse foreground
Smooth background

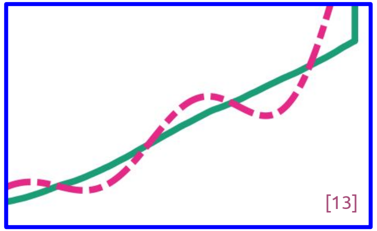
$$\mathbf{y} = \mathbf{\Phi}(s_1 + s_2) + \mathbf{n}$$

$$\underset{s_1, s_2}{\arg\min} \ \frac{1}{2}\|\mathbf{y} - \mathbf{\Phi}(s_1 + s_2)\|_2^2 + \lambda_1 \|\mathrm{L}_1(s_1)\|_{\mathcal{M}} + \frac{\lambda_2}{2}\|\mathrm{L}_2(s_2)\|_{L_2}^2$$

Representer theorem[6] **:**

- $s_1^* \rightarrow$ Sparse spline
- $s_2^* \rightarrow$ Smooth quadratic solution





[13] Debarre T et al."Continuous-Domain Formulation of Inverse Problems for Composite Sparse-Plus-Smooth Signals", *IEEE Open Journal of Signal Processing*, 2021.
[14] Unser M et al., "Splines Are Universal Solutions of Linear Inverse Problems with Generalized TV Regularization", *SIAM Review*, 2017.

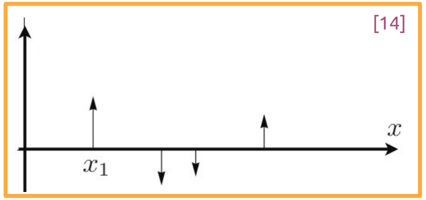
Composite Sparse-plus-Smooth Problems

Sparse foreground
Smooth background

$$\mathbf{y} = \mathbf{\Phi}(s_1 + s_2) + \mathbf{n}$$

$$\arg\min_{s_1, s_2} \frac{1}{2}\|\mathbf{y} - \mathbf{\Phi}(s_1 + s_2)\|_2^2 + \lambda_1 \| \quad s_1\|_{\mathcal{M}} + \frac{\lambda_2}{2}\| \quad s_2\|_{L_2}^2$$

Representer theorem[6] **:**



$$\widehat{s}_1 = m[\widehat{\mathbf{a}}, \widehat{\mathbf{x}}]$$

- s_1^* $\rightarrow$ Sparse **measure**
- s_2^* $\rightarrow$ Smooth quadratic solution

$$\widehat{s}_2 = \mathbf{\Phi}^* \widehat{\mathbf{u}}$$

[13] Debarre T et al."Continuous-Domain Formulation of Inverse Problems for Composite Sparse-Plus-Smooth Signals", *IEEE Open Journal of Signal Processing*, 2021.
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Our Decoupling Representer Theorem

$$\underset{s_1, s_2 \in \mathcal{B} \times \mathcal{H}}{\arg\min} \; \frac{1}{2}\|\mathbf{y} - \mathbf{\Phi}(s_1 + s_2)\|_2^2 + \lambda_1 \boxed{\|s_1\|_{\mathcal{B}}} + \frac{\lambda_2}{2} \boxed{\|s_2\|_{\mathcal{H}}^2}$$

[Theorem 6.1]

$$\boxed{\begin{cases} \widehat{s}_1 \in \underset{s_1 \in \mathcal{B}}{\arg\min} \; \frac{1}{2}\|\mathbf{M}_{\lambda_2}^{-\frac{1}{2}}(\mathbf{y} - \mathbf{\Phi}(s_1))\|_2^2 + \lambda_1 \|s_1\|_{\mathcal{B}} \\ \widehat{s}_2 = \frac{1}{\lambda_2} \mathbf{\Phi}^* \mathbf{M}_{\lambda_2}^{-1}(\mathbf{y} - \mathbf{w}) \end{cases}}$$

$$\mathbf{M}_{\lambda_2} := \frac{1}{\lambda_2}(\mathbf{\Phi}\mathbf{\Phi}^* + \lambda_2 \mathbf{I}_L)$$

$$\mathbf{w} = \mathbf{\Phi}(\widehat{s}_1)$$

Consequences:

→ Decoupled reconstruction procedure

→ Scaling of regularization parameters

Decoupling for Hilbert-plus-Banach Problems

$$\mathbf{M}_{\lambda_2} := \frac{1}{\lambda_2}\left(\mathbf{\Phi}\mathbf{\Phi}^* + \lambda_2 \mathbf{I}_L\right)$$

Theorem 6.1. *Let* $\boldsymbol{y} \in \mathbb{R}^L$, $\lambda_1, \lambda_2 > 0$, $\mathbf{\Phi} \in (\mathcal{A} \cap \mathcal{H})^L$. *Then, the solution set* $\mathcal{W}(\lambda_1, \lambda_2)$ *is non-empty, convex, and weak*-compact in* $\mathcal{B} \times \mathcal{H}$. *Moreover, we can write*

$$\mathcal{W}(\lambda_1, \lambda_2) = \mathcal{V}(\mathbf{M}_{\lambda_2}, \lambda_1) \times \{\hat{f}_2\} \tag{6.14}$$

with

$$\mathcal{V}(\mathbf{M}_{\lambda_2}, \lambda_1) = \underset{s_1 \in \mathcal{B}}{\arg\min} \quad \|\mathbf{M}_{\lambda_2}^{-\frac{1}{2}}(\boldsymbol{y} - \mathbf{\Phi}(s_1))\|_2^2 + \lambda_1 \|s_1\|_{\mathcal{B}}, \tag{6.15}$$

$$\widehat{s}_2 = \frac{1}{\lambda_2}\mathbf{\Phi}^* \mathbf{M}_{\lambda_2}^{-1}\left(\boldsymbol{y} - \boldsymbol{w}\right), \tag{6.16}$$

where the vector $\boldsymbol{w} = \mathbf{\Phi}(\widehat{s}_1)$ *is unique and independent of the solution* $\widehat{s}_1 \in \mathcal{V}(\mathbf{M}_{\lambda_2}, \lambda_1)$.

Consequences:

- Decoupled reconstruction procedure
- Scaling of regularization parameters

Scaling of Regularization Parameters

Proposition 6.3 (Maximum value of λ_1). *Let $\mathcal{X}$ be a continuous domain $\mathcal{X} = \mathbb{R}^d$ or $\mathcal{X} = \mathbb{T}^d$ for $d \in \mathbb{N}^*$.*

We consider the composite optimization problem (6.12) where $\mathcal{B} = \mathcal{M}(\mathcal{X})$ and $\|\cdot\|_{\mathcal{B}} = \|\cdot\|_{\mathcal{M}}$. We define

$$\boxed{\begin{aligned}\lambda_{1,\max} &= \|\mathbf{\Phi}^* \mathbf{M}_{\lambda_2}^{-1} \boldsymbol{y}\|_\infty \\ &= \lambda_2 \|\mathbf{\Phi}^* \left(\mathbf{\Phi}\mathbf{\Phi}^* + \lambda_2 \mathbf{I}_L\right)^{-1} \boldsymbol{y}\|_\infty.\end{aligned}} \tag{6.18}$$

For any $\lambda_1 \geq \lambda_{1,\max}$, the solution set for the Banach component is reduced to the singleton zero

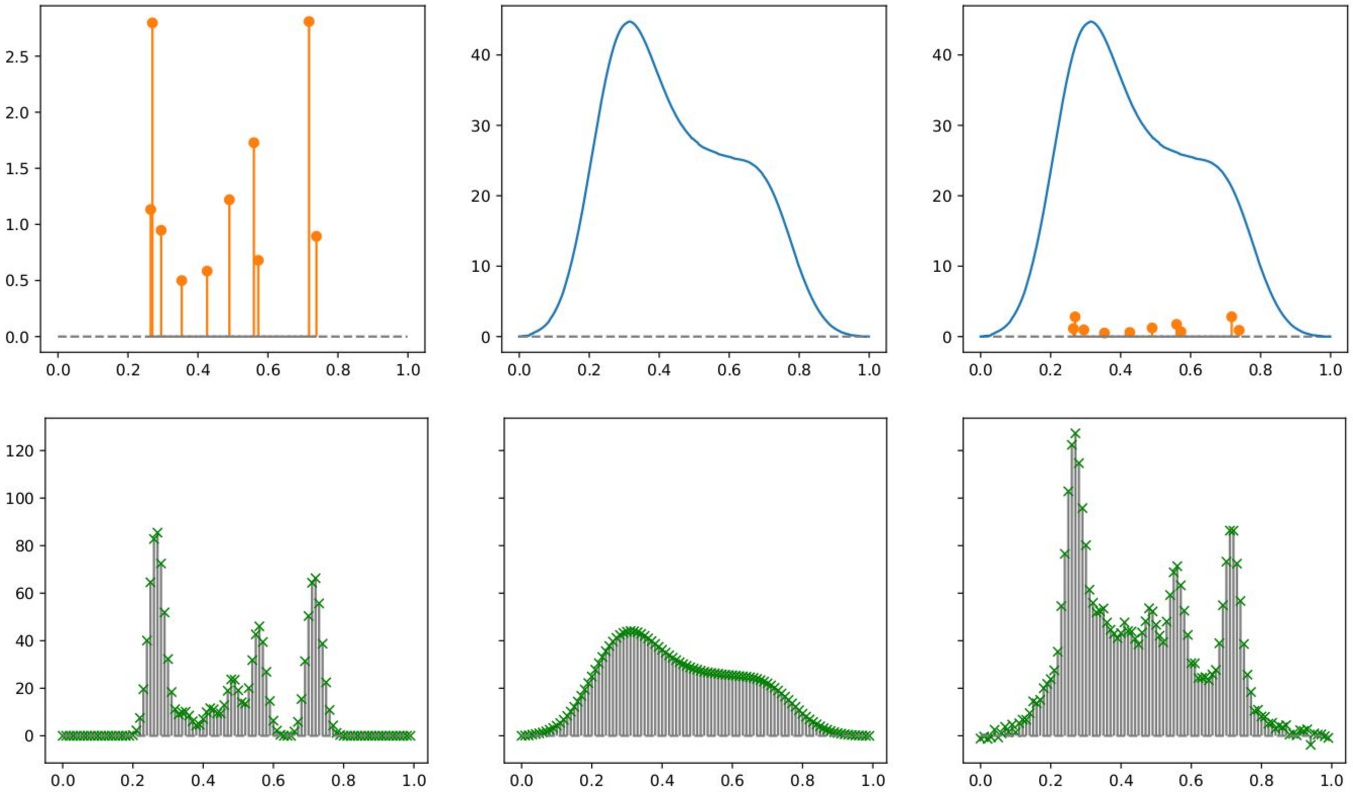
$$\mathcal{V}(\mathbf{M}_{\lambda_2}, \lambda_1) = \{0\}$$

and Problem (6.12) is equivalent to a single-component Hilbert problem.

Simple reconstruction - Simulation

Ground truth

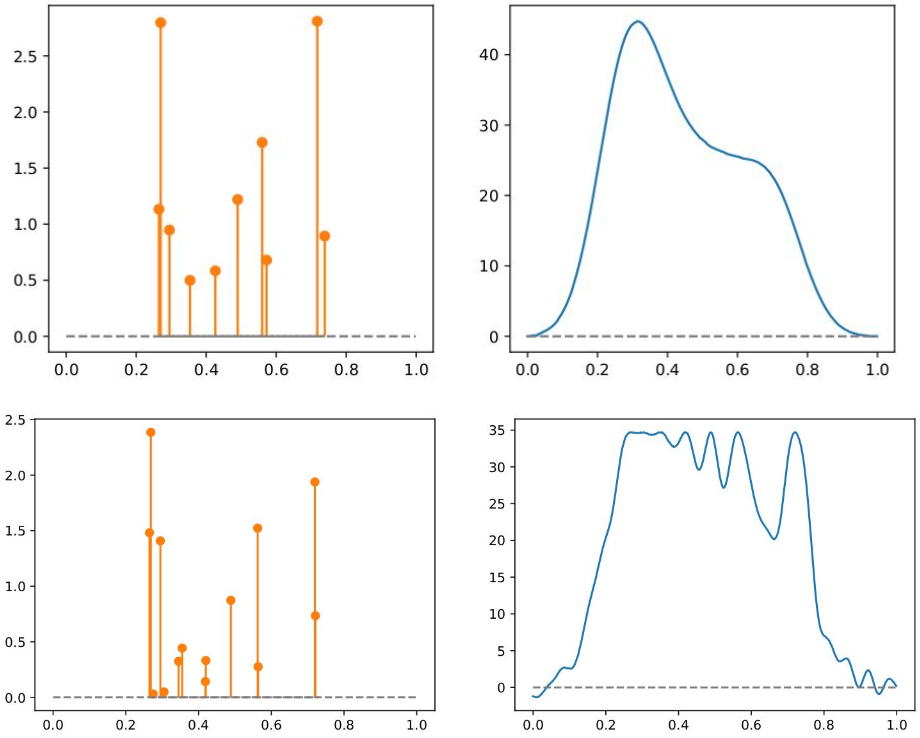
Measurements



Simple reconstruction - Results

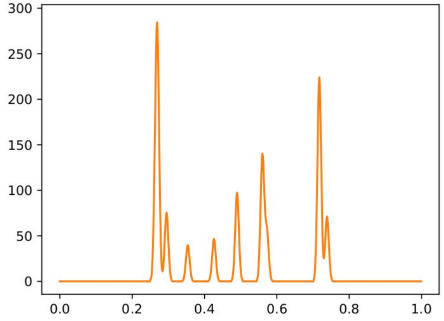
Ground truth

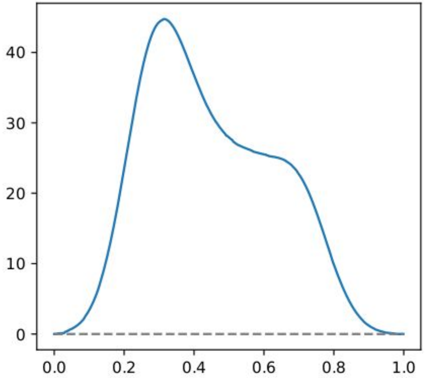
Reconstruction



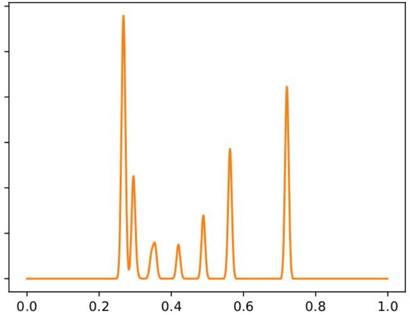
Simple reconstruction - Results

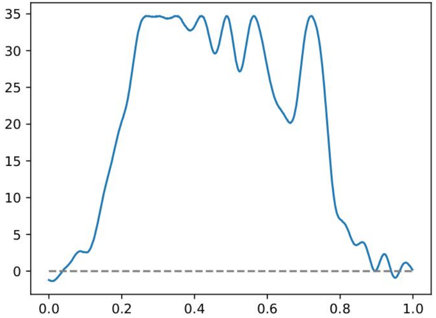
Ground truth





Reconstruction

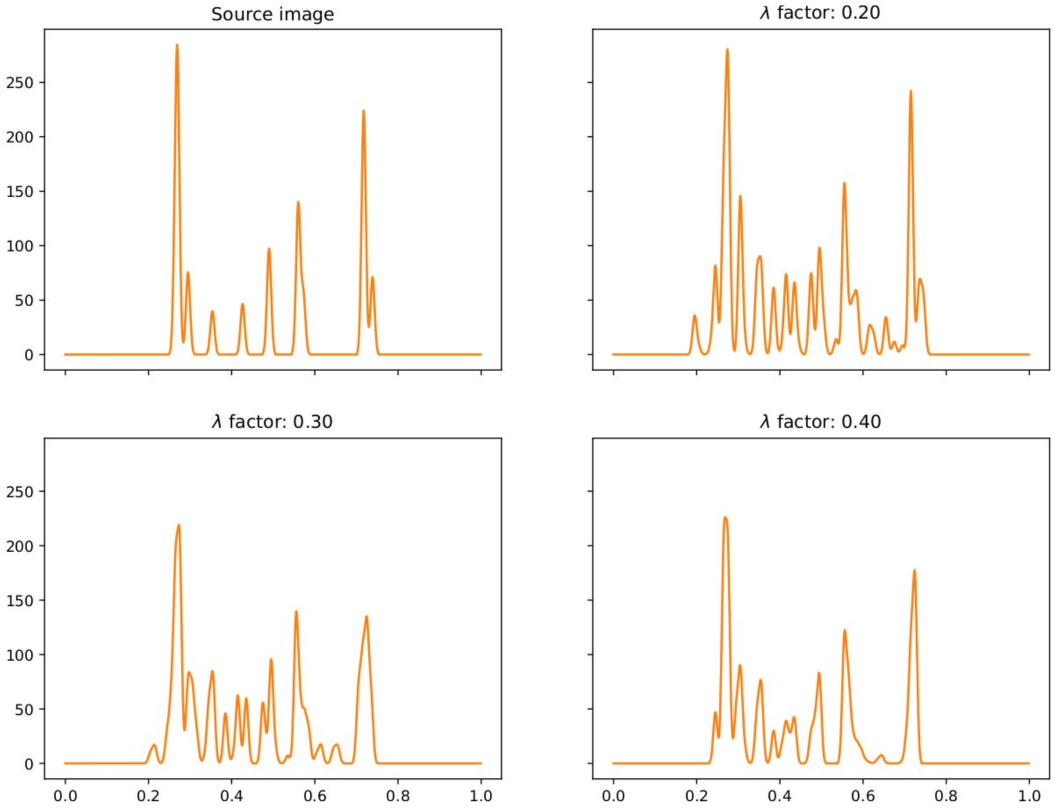




Comparison with single-component reconstruction

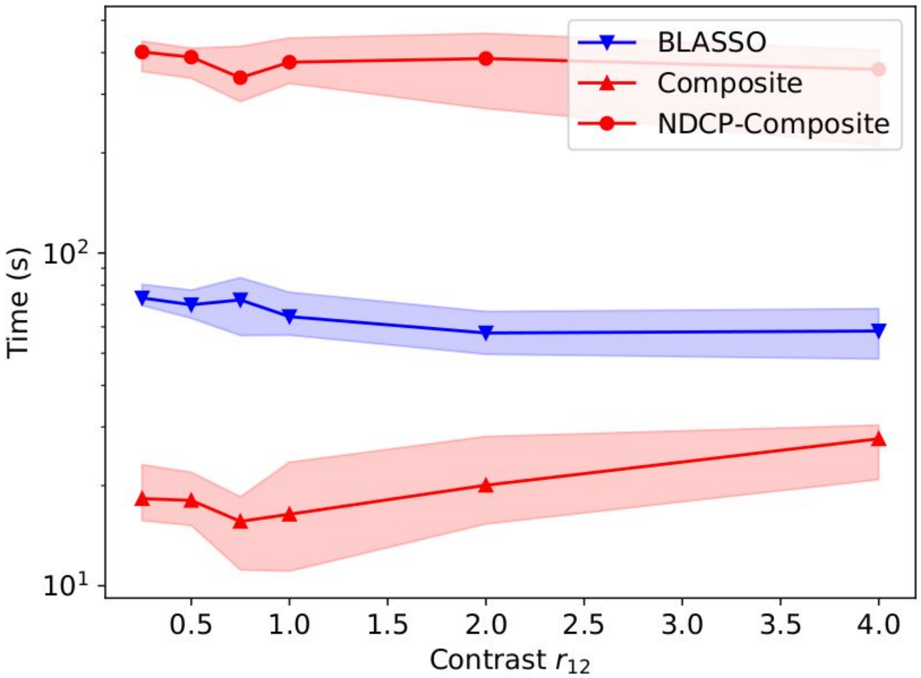
$$\underset{s_1 \in \mathcal{B}}{\arg\min} \left\{ \frac{1}{2}\|\mathbf{y} - \mathbf{\Phi}(s_1)\|_2^2 + \lambda_1 \|s_1\|_{\mathcal{B}} \right\}$$

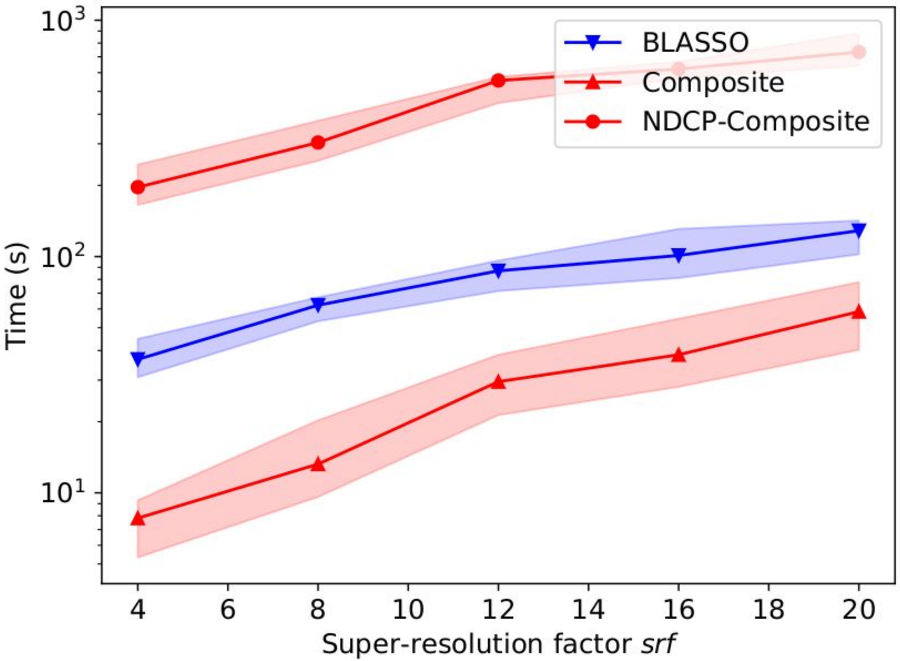
$$\lambda_1 = \lambda_f \cdot \lambda_{1,\max}$$



Comparison with non-decoupled solving

$$r_{1/2} = \frac{\|\mathbf{\Phi}(s_1^\dagger)\|_2}{\|\mathbf{\Phi}(s_2^\dagger)\|_2}$$





Bonus: Discrete Problems with Operators

$$\boxed{\arg\min_{\mathbf{x}_1,\mathbf{x}_2\in\mathbb{R}^N} \frac{1}{2}\left\|\mathbf{y}-\mathbf{A}(\mathbf{x}_1+\mathbf{x}_2)\right\|_2^2 + \lambda_1\left\|\mathbf{L}_1\mathbf{x}_1\right\|_1 + \frac{\lambda_2}{2}\left\|\mathbf{L}_2\mathbf{x}_2\right\|_2^2}$$

Theorem 1[8] (RT for the composite problem ($\boxed{P_{12}}$)). *Under Assumptions* $\boxed{1}$, $\boxed{2}$ *and* $\boxed{3}$, *the solution set* $\mathcal{V}$ *of* ($\boxed{P_{12}}$) *can be written as the Cartesian product*

$$\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$$

where :

1) *The sparse variable* $\mathbf{x}_1$ *belongs to the set* $\mathcal{V}_1$ *defined as*

$$\mathcal{V}_1 = \arg\min_{\mathbf{x}_1\in\mathbb{R}^N} \left\{ \begin{array}{r} \frac{1}{2}(\mathbf{y}-\mathbf{A}\mathbf{x}_1)^T\mathbf{M}_{\lambda_2}(\mathbf{y}-\mathbf{A}\mathbf{x}_1) \\ +\lambda_1\left\|\mathbf{L}_1\mathbf{x}_1\right\|_1 \end{array} \right\} \quad (P_1)$$

with $\mathbf{M}_{\lambda_2} = \lambda_2\mathbf{\Lambda_2}\left(\mathbf{A}\mathbf{A}^T + \lambda_2\mathbf{\Lambda_2}\right)^{-1}$;

2) *All the sparse component solutions share the same measurement vector, that is there exists* $\tilde{\mathbf{y}} \in \mathbb{C}^L$ *such that any* $\mathbf{x}_1^* \in \mathcal{V}_1$ *satisfies* $\mathbf{A}\mathbf{x}_1^* = \tilde{\mathbf{y}}$;

3) *The smooth component solution is unique and independent of the sparse component, so that* $\mathcal{V}_2 = \{\mathbf{x}_2^*\}$. $\mathbf{x}_2^*$ *is the unique solution of the minimization problem*

$$\arg\min_{\mathbf{x}_2\in\mathbb{R}^N} \frac{1}{2}\left\|\mathbf{y}-\tilde{\mathbf{y}}-\mathbf{A}\mathbf{x}_2\right\|_2^2 + \frac{\lambda_2}{2}\left\|\mathbf{L}_2\mathbf{x}_2\right\|_2^2, \quad (P_2)$$

given by $\mathbf{x}_2^* = \mathbf{A}^T\left(\mathbf{A}\mathbf{A}^T + \lambda_2\mathbf{\Lambda_2}\right)^{-1}(\mathbf{y}-\tilde{\mathbf{y}})$.

[8] **Jarret A** et al. "A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization", *32nd European Signal Processing Conference (EUSIPCO)*, 2024

Bonus: Discrete Problems with Operators

$$\underset{\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N}{\arg\min} \frac{1}{2} \left\| \mathbf{y} - \mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) \right\|_2^2 + \lambda_1 \left\| \mathbf{L}_1 \mathbf{x}_1 \right\|_1 + \frac{\lambda_2}{2} \left\| \mathbf{L}_2 \mathbf{x}_2 \right\|_2^2$$

Assumption 1. *The forward matrix* $\mathbf{A} \in \mathbb{R}^{L \times N}$ *is surjective, i.e., has full row rank, so that* $\mathbf{A}\mathbf{A}^T$ *is invertible.*

Assumption 2. *The nullspaces of the forward matrix and the regularization matrix* $\mathbf{L}_2 \in \mathbb{R}^{M_2 \times N}$ *have a trivial intersection, that is* $\ker \mathbf{A} \cap \ker \mathbf{L}_2 = \{\mathbf{0}\}$.

Assumption 3. *The vector space* $\ker(\mathbf{A})^\perp$ *is an invariant subspace of the operation* $\mathbf{L}_2^T \mathbf{L}_2$*, i.e., the following holds:* $\mathbf{x} \in \ker(\mathbf{A})^\perp \Rightarrow \mathbf{L}_2^T \mathbf{L}_2 \mathbf{x} \in \ker(\mathbf{A})^\perp$.

Under Assumption $\boxed{1}$, we define the matrix $\mathbf{\Lambda_2} \in \mathbb{R}^{L \times L}$ as

$$\mathbf{\Lambda_2} = \left(\mathbf{A}\mathbf{A}^T\right)^{-1} \mathbf{A} \mathbf{L}_2^T \mathbf{L}_2 \mathbf{A}^T. \qquad (1)$$

[8] **Jarret A** et al. "A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization", *32nd European Signal Processing Conference (EUSIPCO)*, 2024