To Grid or Not To Grid

Atomic Methods for Sparse Inverse Problems

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under the direction of
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November 7th, 2025

Part I: What is it about?

1. Linear Inverse Problems
2. Sparse Signals
3. Penalized Optimization
4. Atomic Reconstruction Methods

"Méthodes atomiques pour la résolution de **problèmes inverses** parcimonieux"

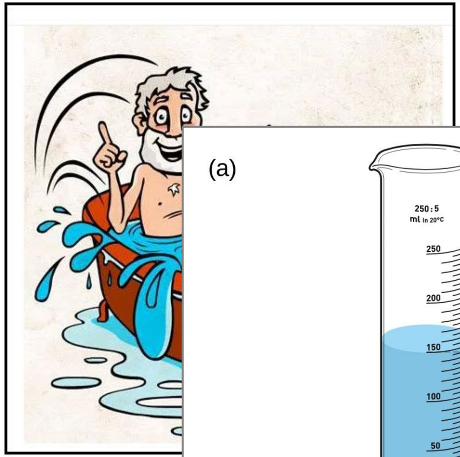
“En science, un **problème inverse** est une situation dans laquelle on tente de déterminer les causes d'un phénomène à partir des observations expérimentales de ses effets.”

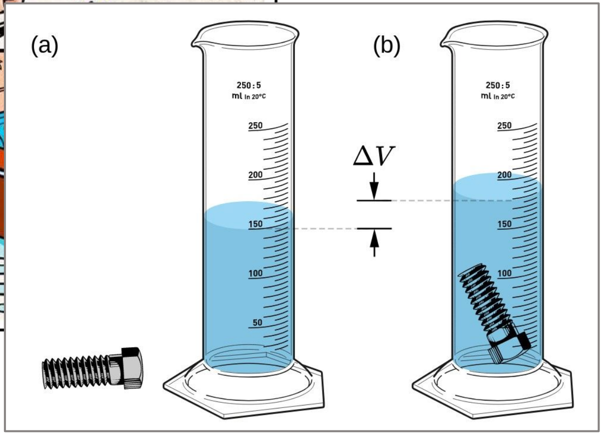
Wikipedia, FR

“An **inverse problem** in science is the process of calculating from a set of observations the causal factors that produced them [...] .”

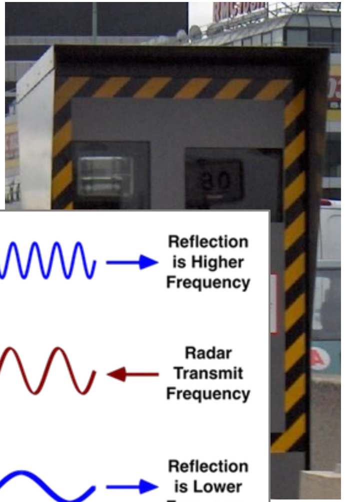
Wikipedia, EN

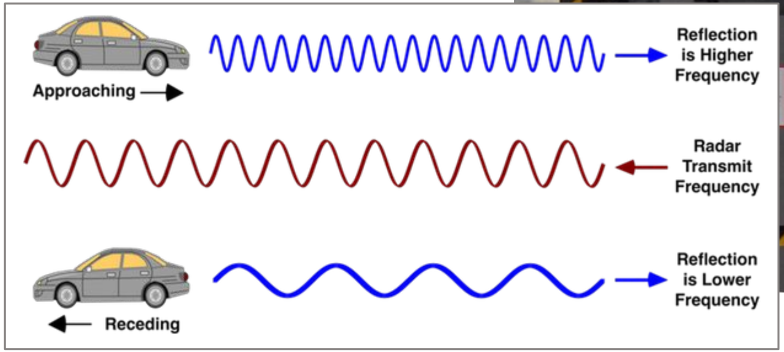
Inverse Problems ?





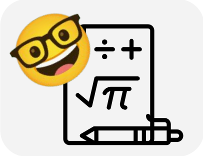
“Doppler effect”





[Credits: Janice VanCleave, handsomescienceteacher.com, Wikimedia, Neltronics]

Maths and formalism



Unknown signal

$$\mathbf{x} \xrightarrow{\mathbf{A}} \mathbf{y}$$

Measurement model

Physical observations

- volume
- speed
- image
- time series
- …

- distance
- frequency
- image
- interferences
- …

"Forward/design equation"

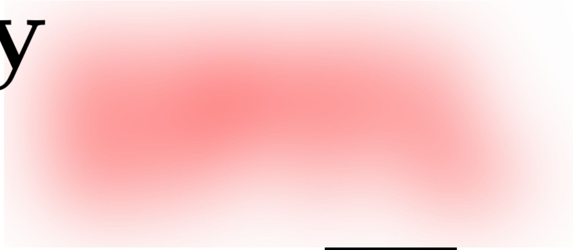
$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Linear Inverse Problems

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

"Deconvolution"

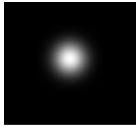
$\mathbf{y}$



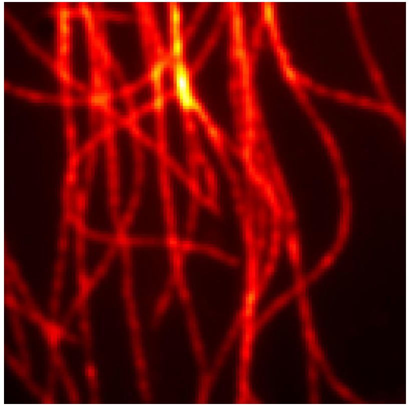
$\mathbf{y}$

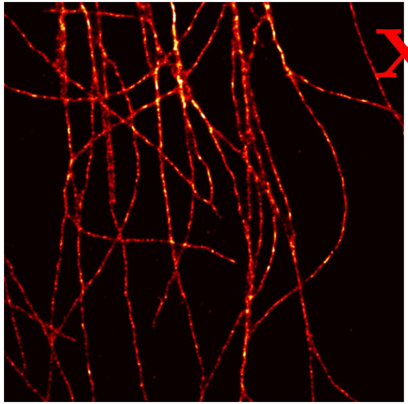
$\mathbf{A}$

$\mathbf{A}$



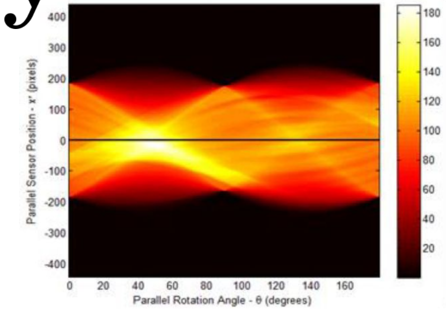
$\mathbf{y}$

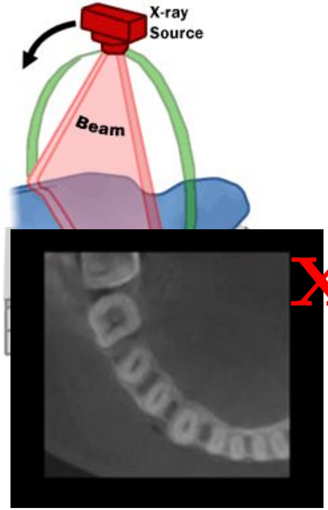




$\mathbf{x}$

$\mathbf{y}$





$\mathbf{x}$

"SMLM"

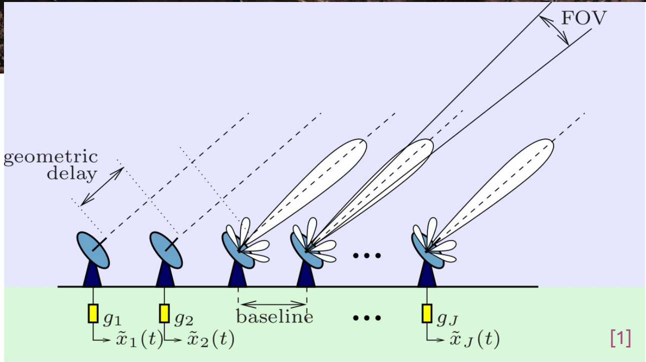
[Credits: Bastien Laville]

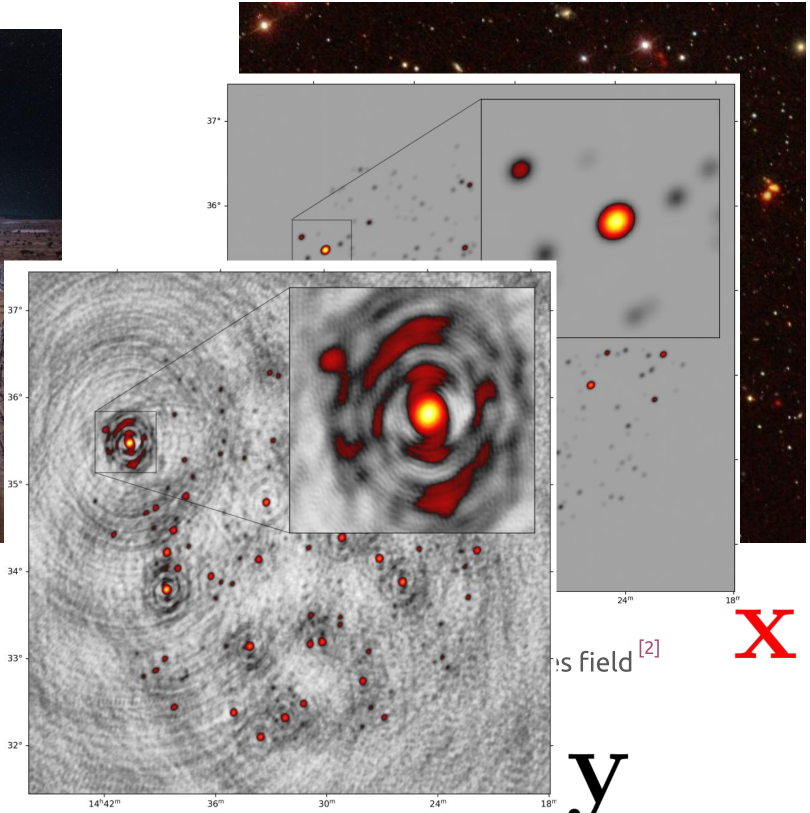
[Credits: Wiipedia: CT Scan]

Radio Interferometric Imaging (still LIP)

[Credits: SKAO]

$\mathbf{A}$





Dirty image

$\mathbf{x}$

$\mathbf{y}$

[1] Van der Veen et al. "Signal Processing for Radio Astronomy", 2019
[2] Williams WL et al., "LOFAR 150-MHz observations of the Boötes field: catalogue and source counts.", *Monthly Notices of the Royal Astronomical Society*, 2016

Ill-Posed Linear Inverse Problems

$$\boxed{\mathbf{y} = \mathbf{A}\mathbf{x}}$$

$$\cancel{\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}}$$

- Measurement noise
- Instability
- Non-uniqueness
- Missing data

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$$

$$\mathbf{A}\mathbf{x}_1 = \mathbf{A}\mathbf{x}_2$$

$$\cancel{\mathbf{A}^{-1}}$$

$\mathbf{y}$

$\mathbf{x}$

??

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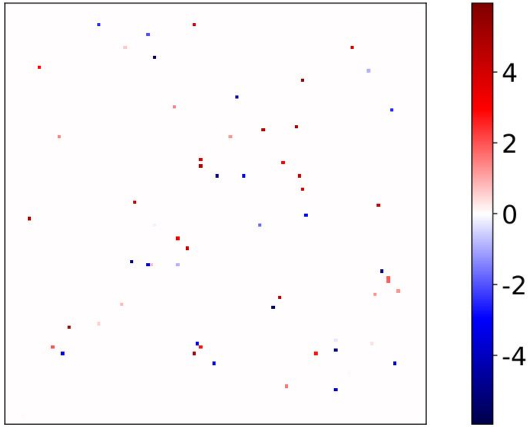
$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

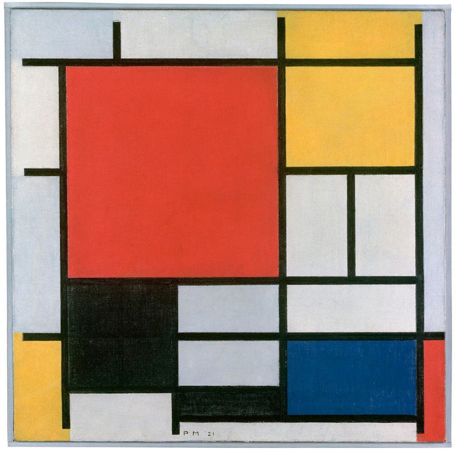
"Méthodes atomiques pour la résolution de problèmes inverses **parcimonieux**"

2. Parcimonie / Sparsity

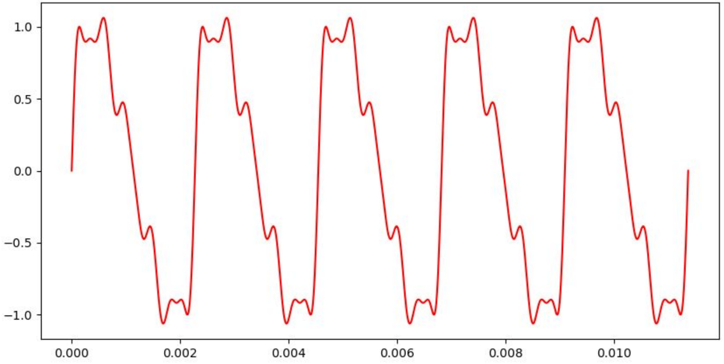


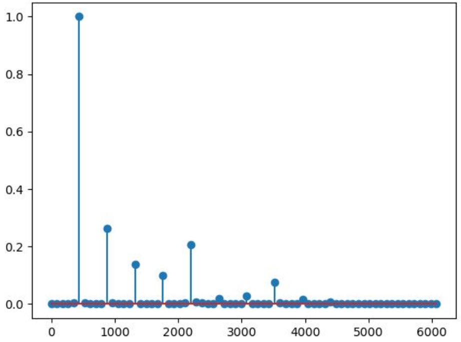
Simulated sparse signal

GLEAM observations of the sky



Composition in Red, Yellow, Blue and Black - Piet Mondrian, 1921





A440 violin

$$\mathbf{x} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \dots + \alpha_K \mathbf{a}_K$$

$$\mathbf{x} = \sum_k \alpha_k \mathbf{a}_k$$

2. (Bonus) All signals can be approx.-sparse

1% information

2. (Bonus) Reconstruction example with sparsity

Something like inpainting or deblurring of an image, with simple sparsity operator

Or Radio Interferomety

Results are maybe not as impressive as what we can see nowadays with AI, but it is controlled, stable and reproducible -> another debate

2. (Bonus) Historical moment

Consequence of this approach: change of paradigm, birth of compressed sensing, reduction of the number of observations

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$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

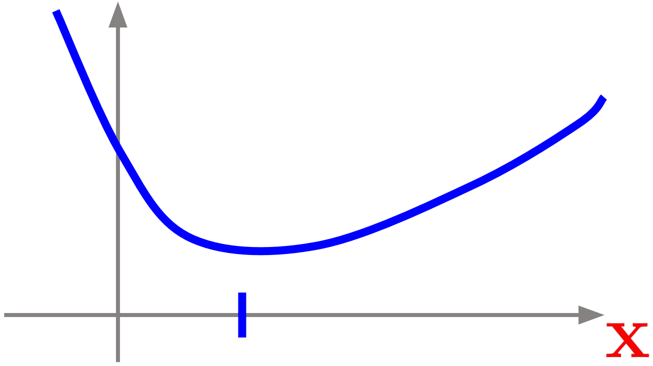
$$\mathbf{x} = \sum \alpha_k \mathbf{a}_k$$

"Méthodes atomiques pour la **résolution** de problèmes inverses parcimonieux"

3. Penalized Optimization

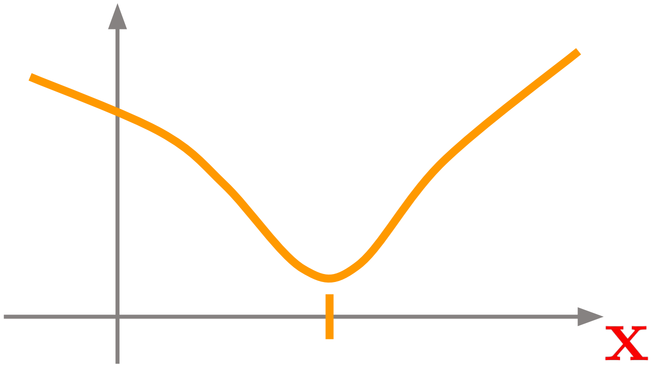
$$\mathbf{y} = \mathbf{A}\mathbf{x} \qquad \mathbf{y} - \mathbf{A}\mathbf{x} = 0$$

$$E(\mathbf{y} - \mathbf{A}\mathbf{x}) \searrow$$



“Data-fidelity”

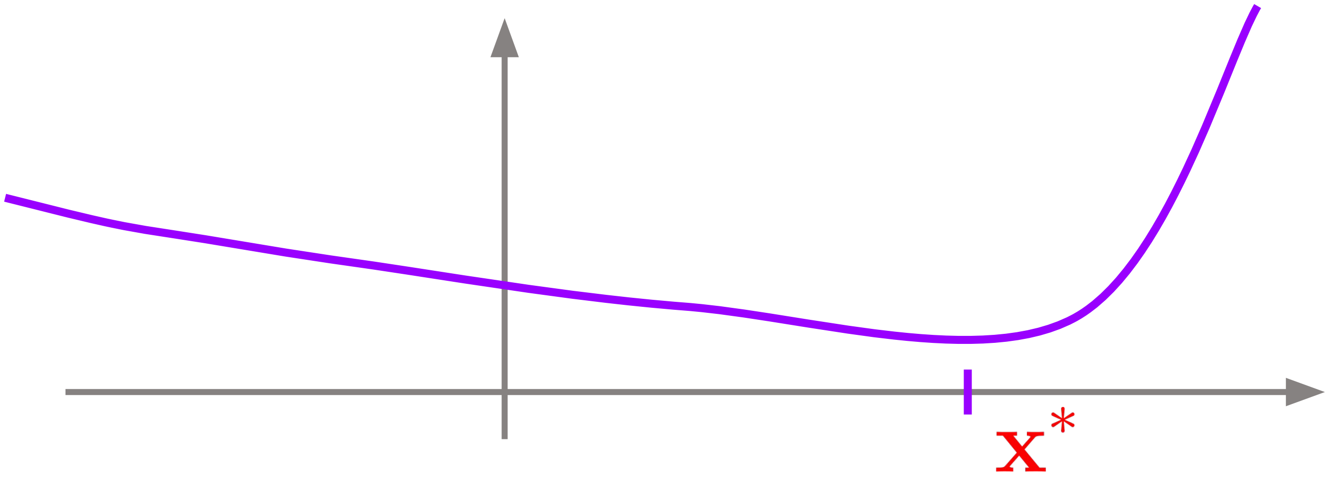
$$R(\mathbf{x}) \searrow$$



“Penalty/Regularization”

3. Penalized Optimization

$$\min_{\mathbf{x}} E(\mathbf{y} - \mathbf{A}\mathbf{x}) + \lambda R(\mathbf{x})$$



LASSO problem[3] : $$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1$$

[3] Tibshirani R. "Regression Shrinkage and Selection via the Lasso", *Journal of the Royal Statistical Society Series B (Methodological)*, 1996.

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1. ~~Linear Inverse Problems~~
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4. Atomic Reconstruction Methods

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

$$\mathbf{x} = \sum \alpha_k \mathbf{a}_k$$

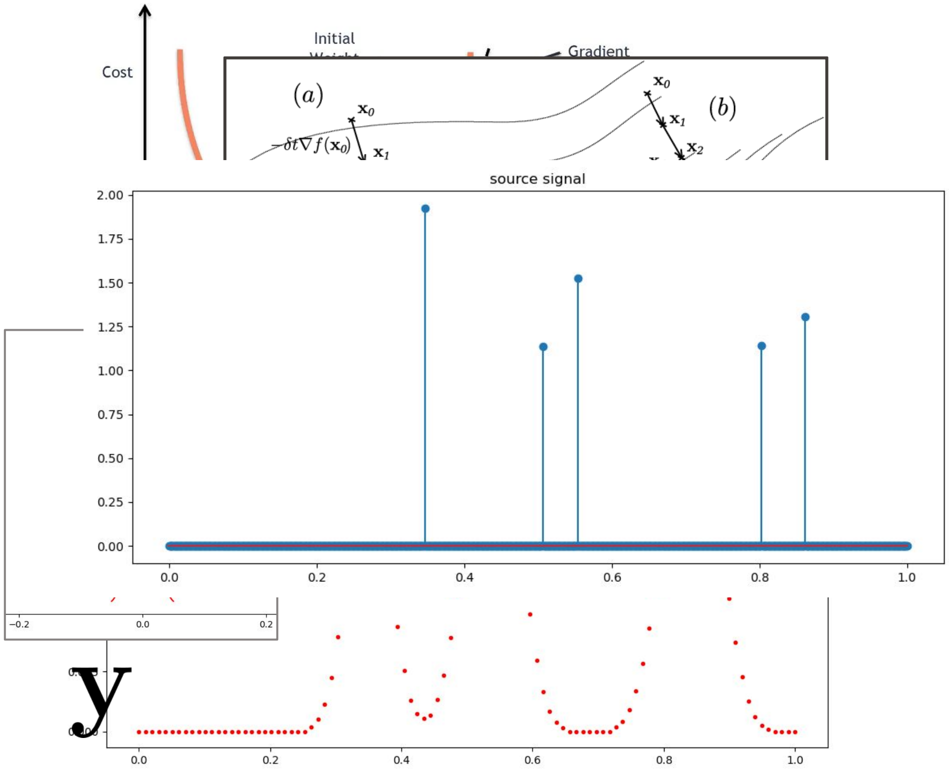
$$\min_{\mathbf{x}} E$$

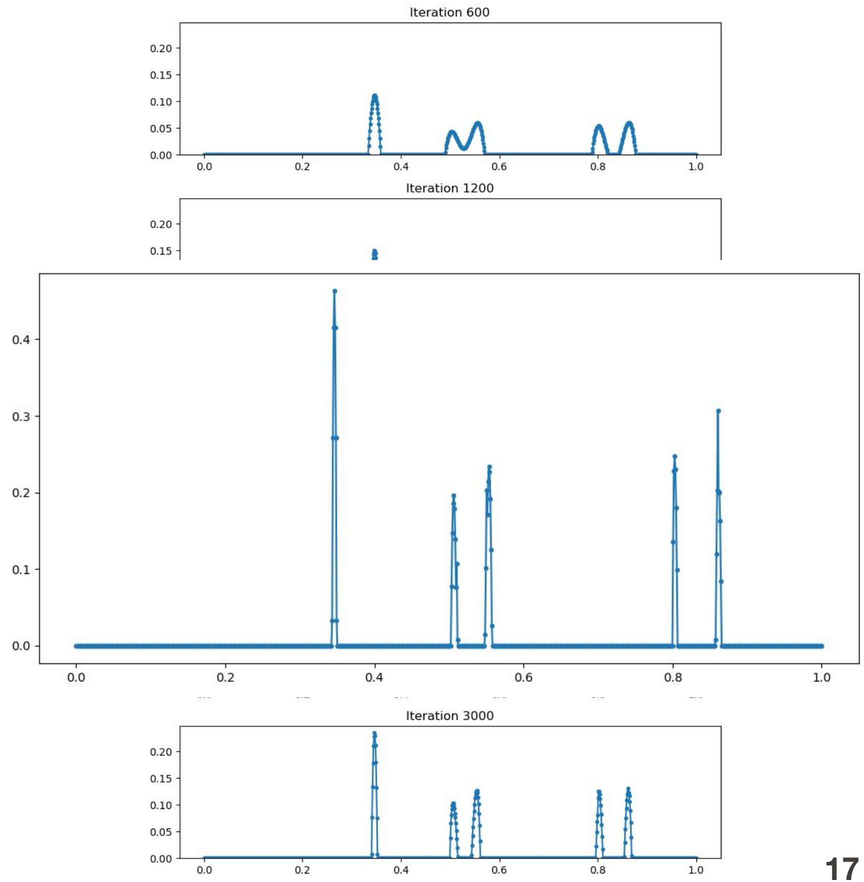
“**Méthodes atomiques** pour la **résolution** de **problèmes inverses parcimonieux**”

4. Numerical reconstruction methods

$$\min_{\mathbf{x}} E(\mathbf{y} - \mathbf{A}\mathbf{x}) + \lambda R(\mathbf{x})$$

"Classical" gradient descent:

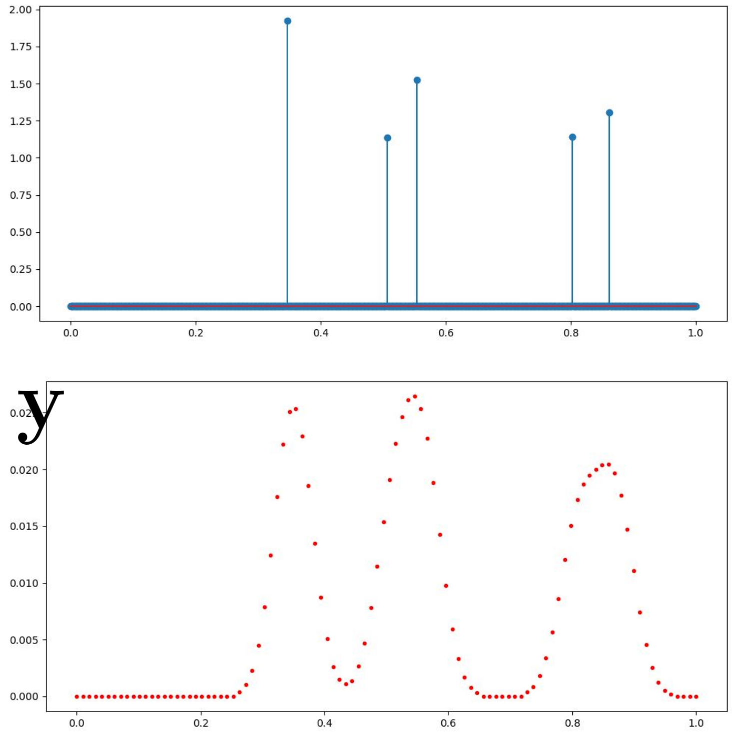


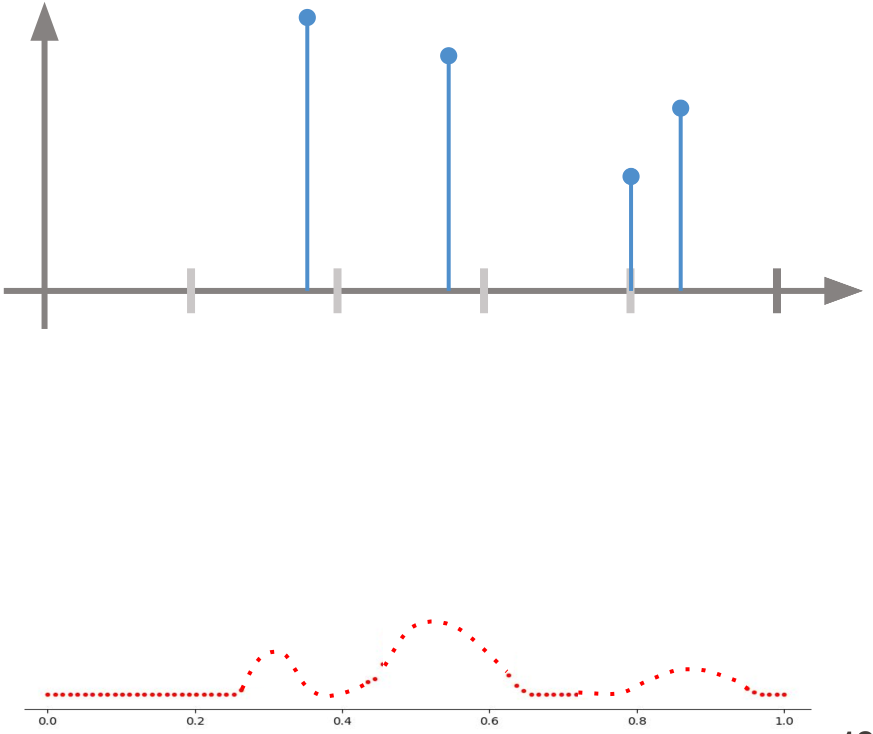


[Credits: www.analyticsvidhya.com, ludovicarnold.com]

4. Numerical reconstruction methods

Atomic method:





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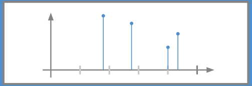
1. ~~Linear Inverse Problems~~
2. ~~Sparse Signals~~
3. ~~Penalized Optimization~~
4. ~~Atomic Reconstruction Methods~~
5. **To Grid or Not To Grid ?**

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

$$\mathbf{x} = \sum \alpha_k \mathbf{a}_k$$

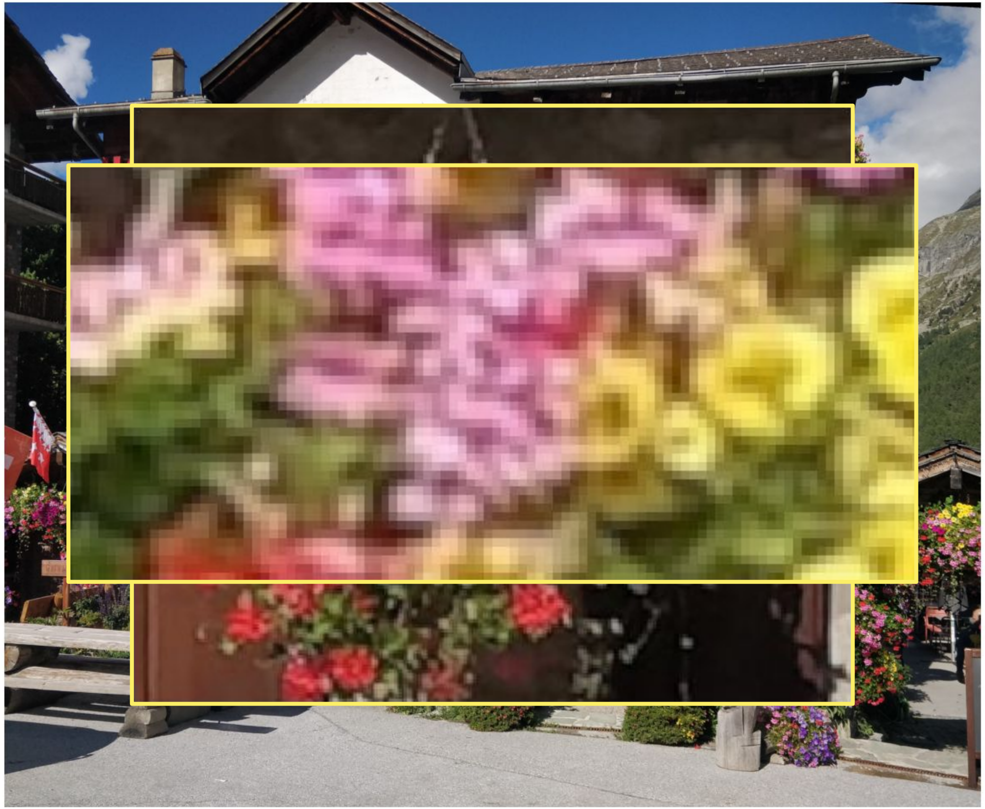
$$\min_{\mathbf{x}} E$$

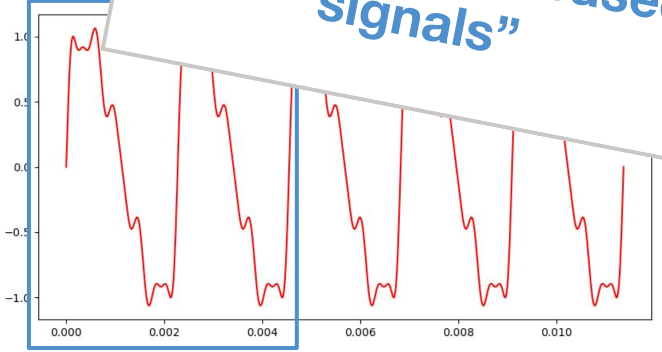


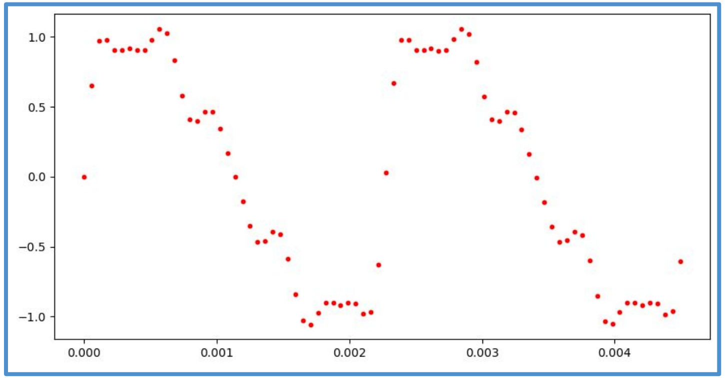
"Méthodes atomiques pour la résolution de problèmes inverses parcimonieux"

5. Continuous-domain reconstructi

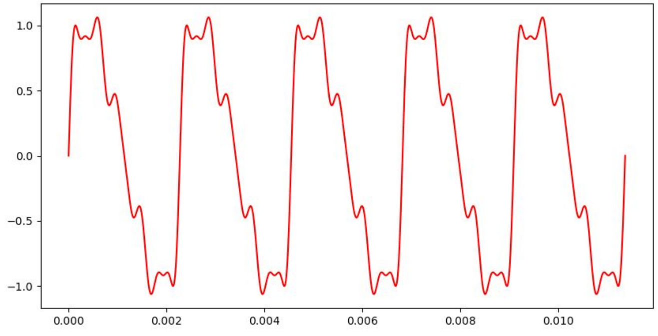
“Discrete/grid-based signals”

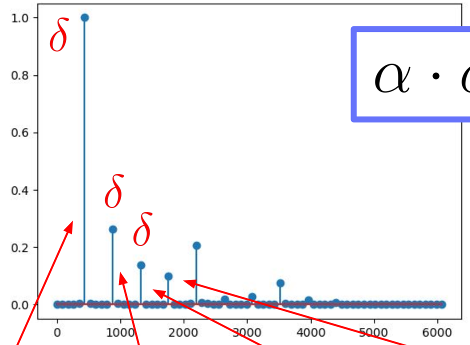






5. Continuous-domain reconstruction?





$\boxed{\alpha \cdot \delta_x}$

…

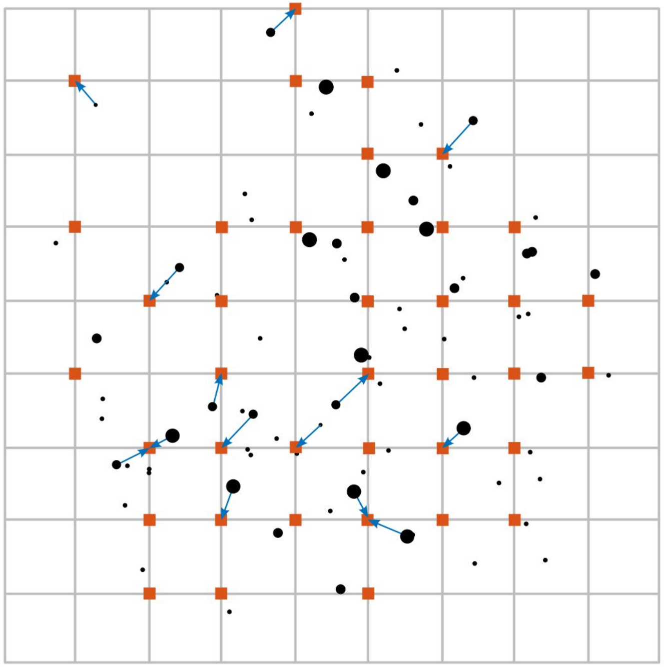
(440 Hz, 1)

(1320 Hz, 0.14)

(880 Hz, 0.26)

$\in \mathcal{M}(\mathbb{R}^d)$

$\mathbf{x} \in \mathbb{R}^N \qquad m = \sum \alpha_k \delta_{x_k}$



[Credits: H. Pan et al., “LEAP”, *A&A*, 2017.]

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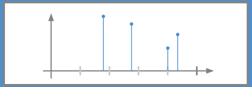
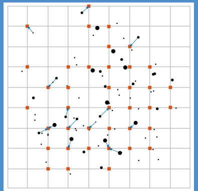
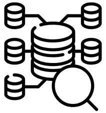
$$\min_{\mathbf{x}} E$$





\+ Radio Interferometry

Challenges:

- Time efficiency 
- Accuracy 
- Scalability 

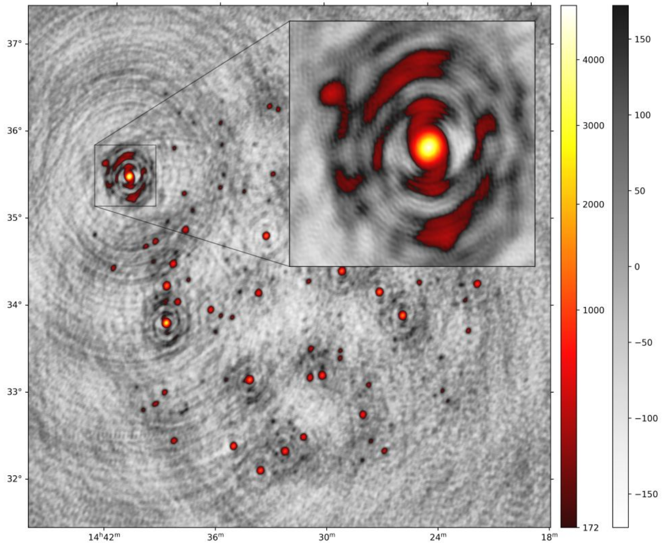
Part II: Focus on key contributions

1. PolyCLEAN: A Polyatomic Method for Radio Interferometry
2. Decoupling of Composite Problems

Radio Interferometric Imaging - Dirty Image

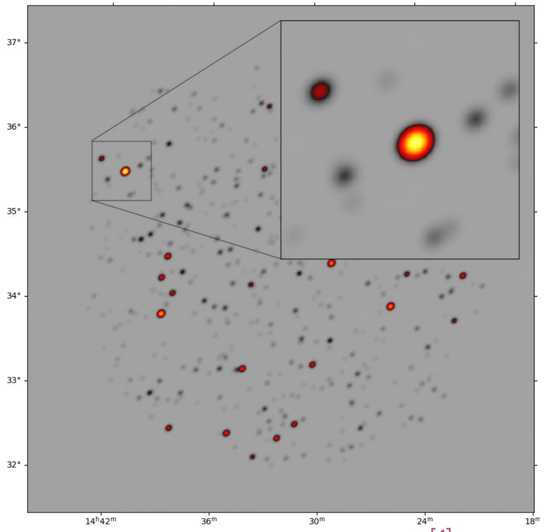
$$\mathbf{V} = \mathbf{\Phi}\mathbf{I} + \varepsilon$$

$$\mathbf{I}_D = \mathbf{\Phi}^*\mathbf{V}$$



Dirty image from observations

$$\mathbf{I}$$



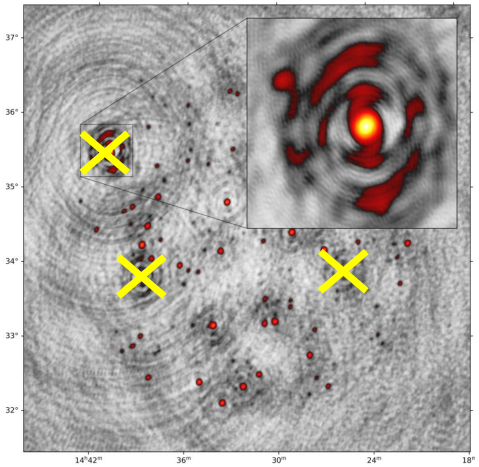
Catalogue Boötes field [4]

[4] Williams WL et al., "LOFAR 150-MHz observations of the Boötes field: catalogue and source counts.", *Monthly Notices of the Royal Astronomical Society*, 2016

Classical Approaches

The CLEAN family[3]

$\mathbf{I}_D$



✅ Intuitive and simple method, long-developed and fast

❌ Sensitive to stop, objective function unclear, physically impossible artefacts

Optimization-based Methods[4]

$$\underset{\mathbf{I}\in\mathbb{R}^N}{\arg\min}\ \frac{1}{2}\|\mathbf{y} - \mathbf{\Phi}\mathbf{I}\|_2^2 + \mathcal{R}(\mathbf{I})$$

✅ Controlled solutions, versatile priors, excellent results, additional tools

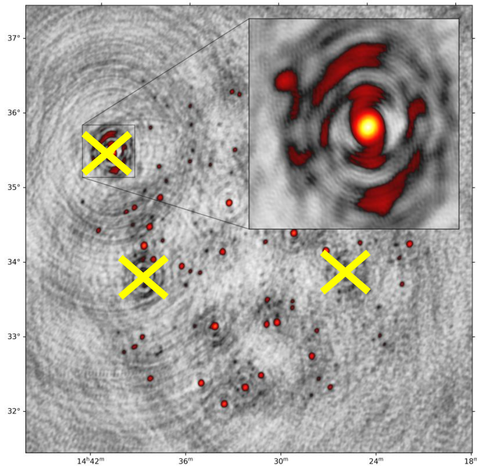
[3] Högbom JA., "Aperture Synthesis with a Non-Regular Distribution of Interferometer Baselines", *Astronomy and Astrophysics Supplement Series*, 1974.

[4] Wiaux Y. et al., "Compressed sensing imaging techniques for radio interferometry", *Monthly Notices of the Royal Astronomical Society*, 2009.

Classical Approaches

The CLEAN family[3]

$\mathbf{I}_D$



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Optimization-based Methods[4]

$$\underset{\mathbf{I}\in\mathbb{R}^N}{\arg\min}\ \frac{1}{2}\|\mathbf{y}-\mathbf{\Phi}\mathbf{I}\|_2^2+\boxed{\lambda\|\mathbf{I}\|_1}$$

LASSO

✅ Controlled solutions, versatile priors, excellent results, additional tools

❌ Numerically heavy, little adoption in the field

[3] Högbom JA., “Aperture Synthesis with a Non-Regular Distribution of Interferometer Baselines”, *Astronomy and Astrophysics Supplement Series*, 1974.

[4] Wiaux Y. et al., “Compressed sensing imaging techniques for radio interferometry”, *Monthly Notices of the Royal Astronomical Society*, 2009.

Our Polyatomic Frank-Wolfe Algorithm

$$\underset{\mathbf{x}\in\mathbb{R}^N}{\arg\min}\ \mathcal{J}(\mathbf{x}) := \frac{1}{2}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1$$

Algorithm 3: Polyatomic Frank-Wolfe algorithm of quality δ [9]

Initialize $\mathbf{x}_0 \in \mathcal{D}$, $\mathcal{S}_0 = \emptyset$

for $k = 1, 2\cdots$ **do**

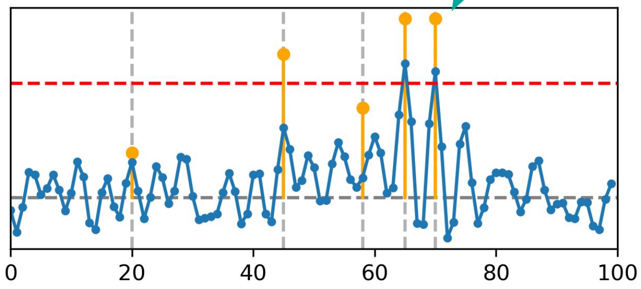
1) Find update directions:

$$\mathcal{I}_k \leftarrow \{1 \leq j \leq N : |\boldsymbol{\eta}_k[j]| \geq \|\boldsymbol{\eta}\|_\infty - \delta/k\}$$

$$\mathcal{S}_k \leftarrow \mathcal{S}_{k-1} \cup \mathcal{I}_k$$

2) Reweight:

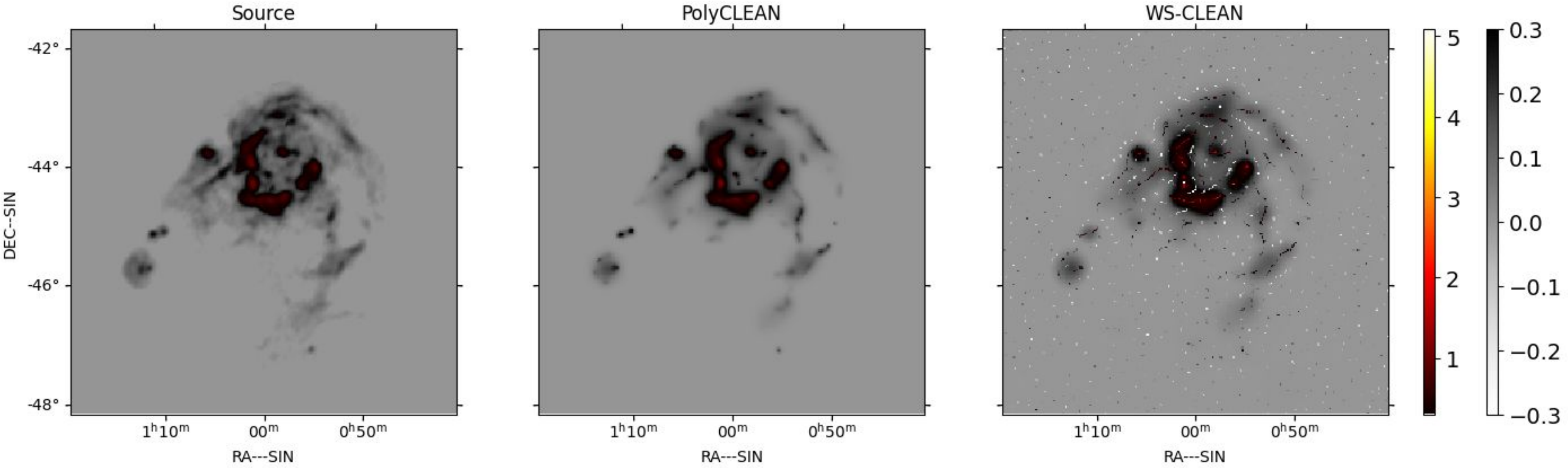
$$\mathbf{x}_k \leftarrow \underset{\mathrm{Supp}(\mathbf{x})\subset\mathcal{S}_k}{\arg\min}\ \frac{1}{2}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2 + \lambda\|\mathbf{x}\|_1$$



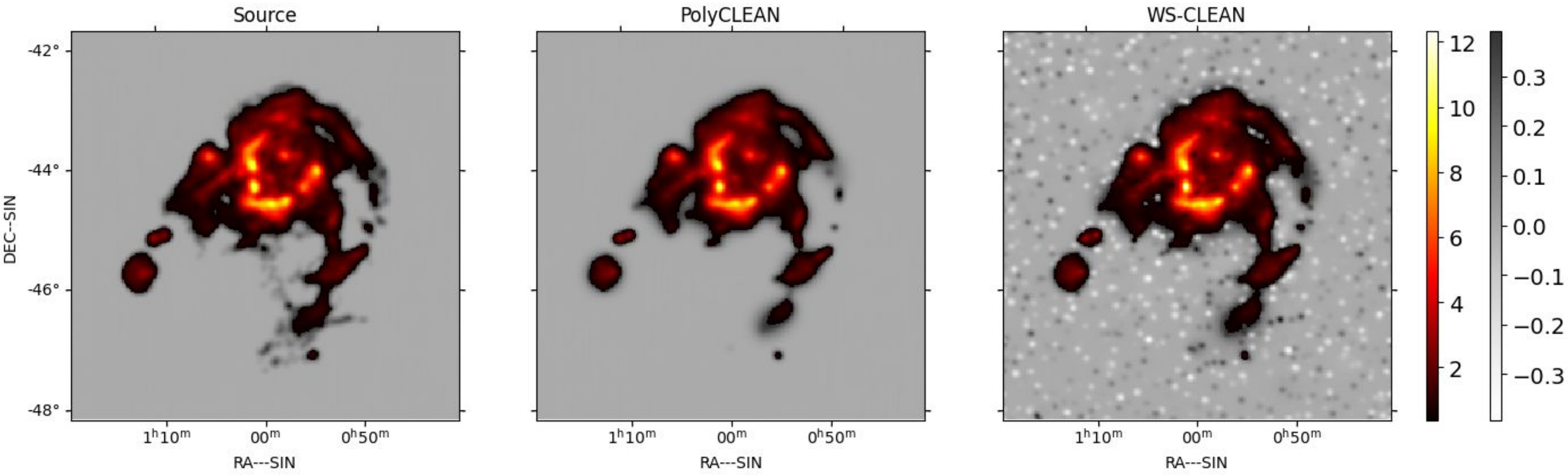
$$|\mathcal{S}_k| \ll N$$

[9] **Jarret A**, Fageot J, Simeoni M. "A Fast and Scalable Polyatomic Frank-Wolfe Algorithm for the LASSO", *IEEE Signal Processing Letters*, 2022.

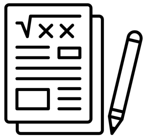
Diffuse emission reconstruction



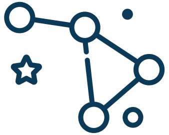
Diffuse emission reconstruction



Summary and Benefits



Design of optimization algorithm → Real world application



Best of both worlds:

- Benefits of CLEAN → Atomic, fast
- Benefits of convex optimization → Accurate
- Sparsity-aware processing (HVOX) → Numerical efficiency
- Question of resolution
  → CLEAN beam is too coarse, certificate beam is data-inspired

EPFL

Adrian Jarret — Public Defense — 7.11.2025

Part II: Focus on key contributions

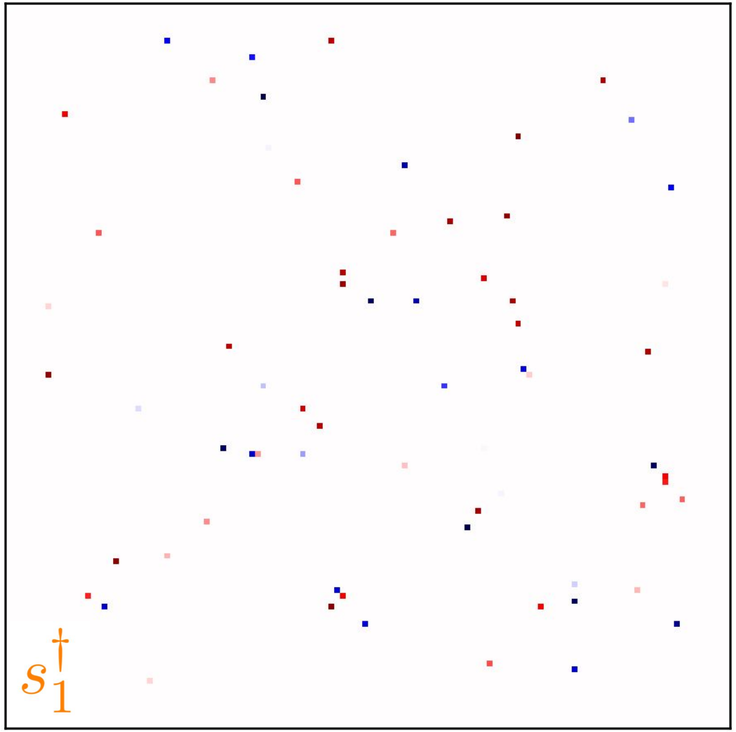
1. ~~PolyCLEAN: A Polyatomic Method for Radio Interferometry~~
2. Decoupling of Composite Problems

Sparse-plus-Smooth Problems

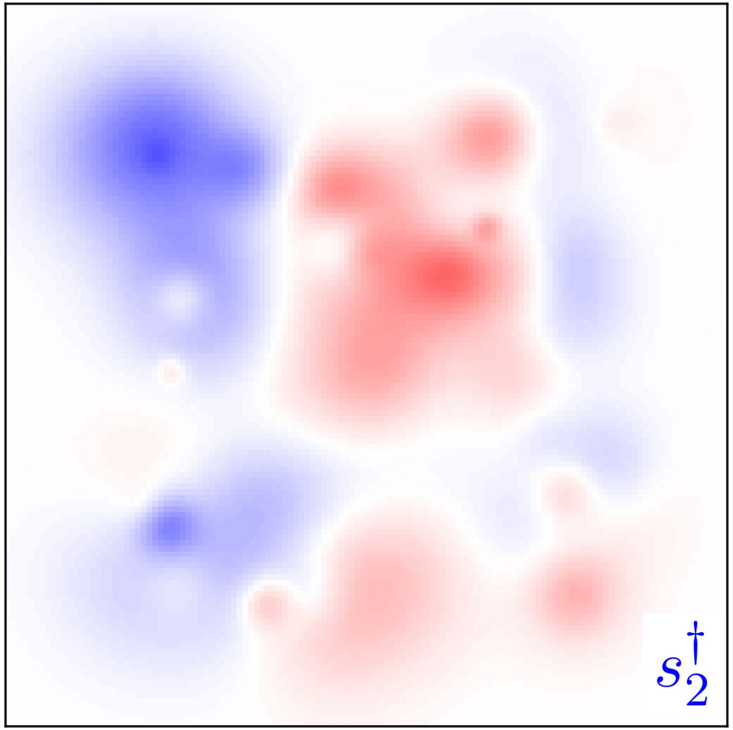
GLEAM survey of the radio sky, J2000 coordinates (9h37min15.21s, 50°25'03.1'')

Sparse-plus-Smooth Problems

$s_1^\dagger + s_2^\dagger$



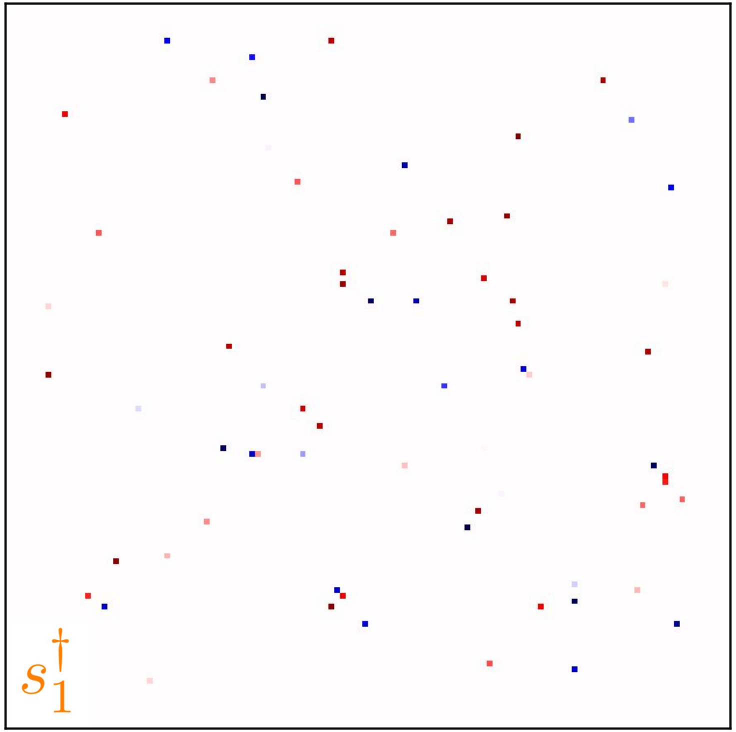
Sparse foreground



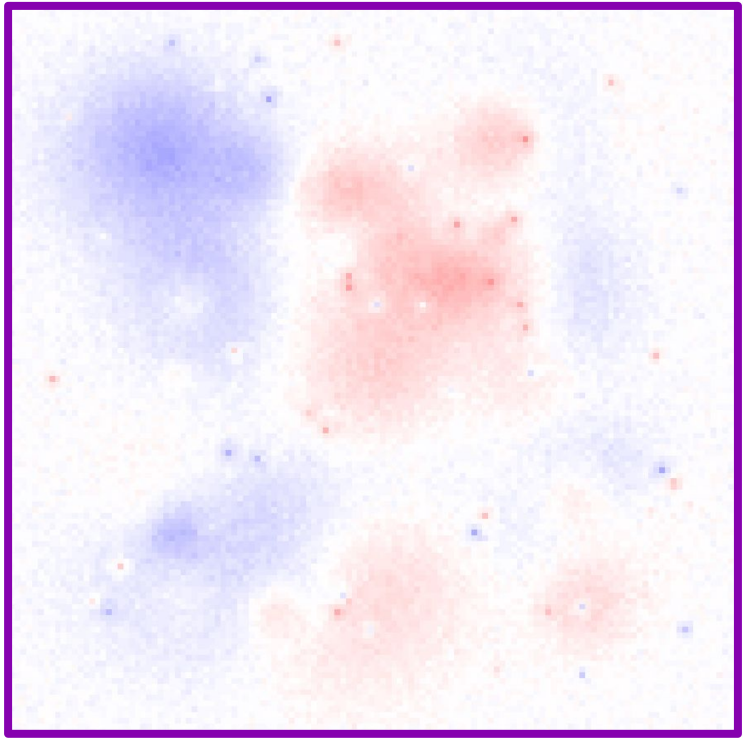
Smooth background

[8] **Jarret A** et al. "A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization", *32nd European Signal Processing Conference (EUSIPCO)*, 2024

Sparse-plus-Smooth Problems



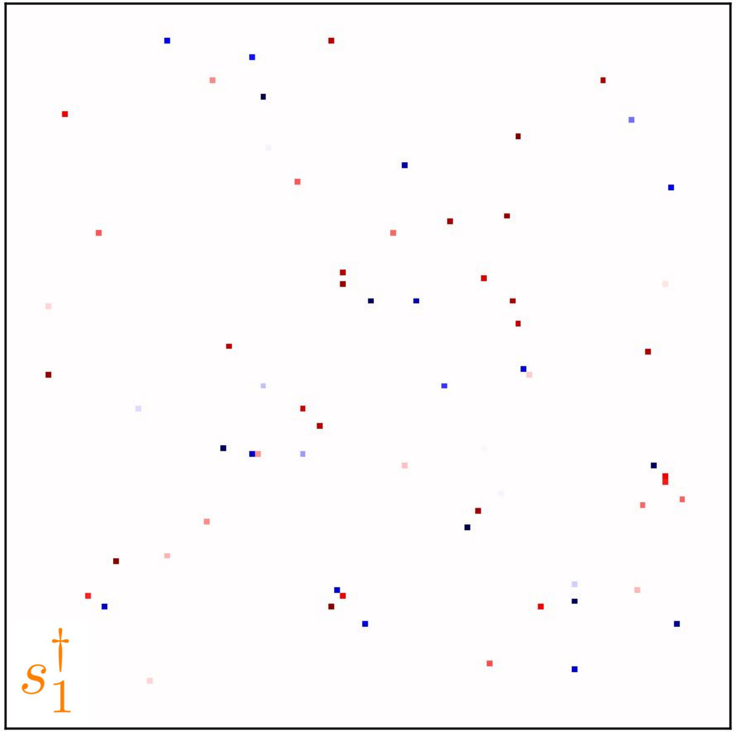
Sparse foreground



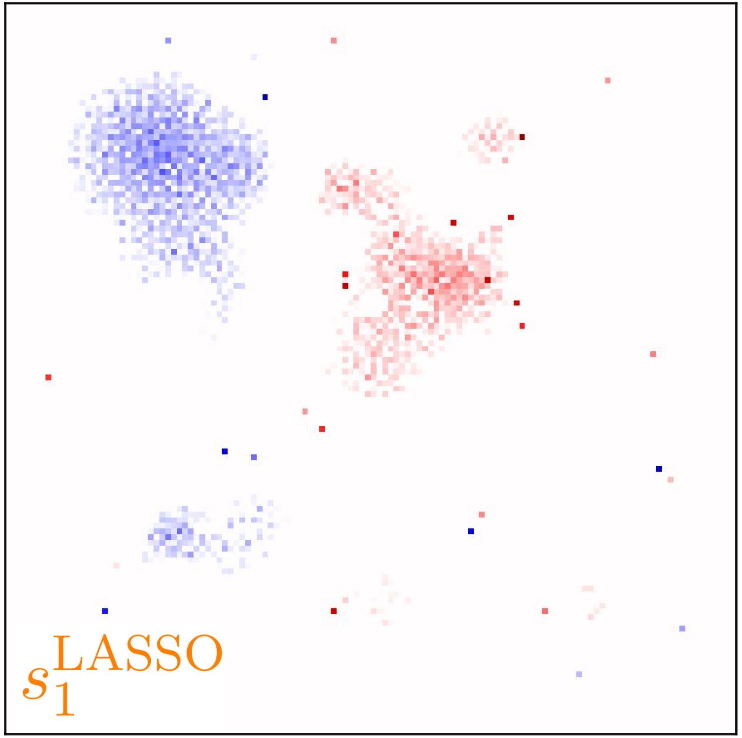
Dirty Image

[8] **Jarret A** et al. "A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization", *32nd European Signal Processing Conference (EUSIPCO)*, 2024

Sparse-plus-Smooth Problems



Sparse foreground

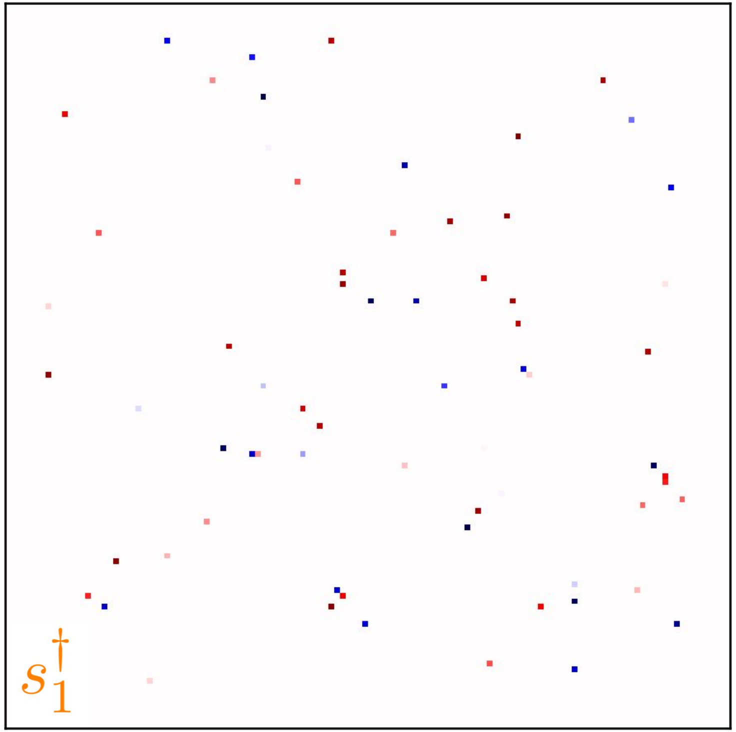


LASSO reconstruction

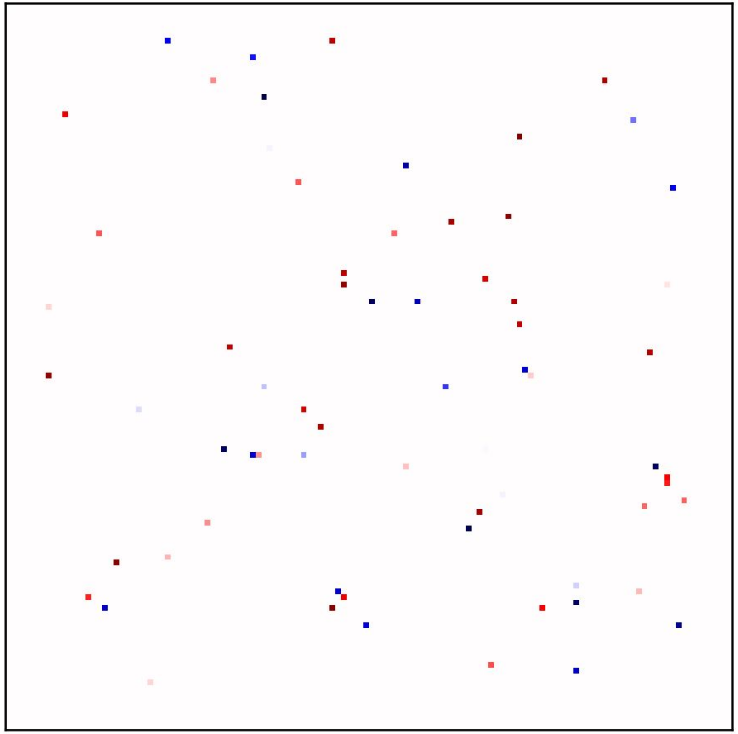
[8] **Jarret A** et al. "A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization", *32nd European Signal Processing Conference (EUSIPCO)*, 2024

Sparse-plus-Smooth Problems

$$\arg\min_{s_1, s_2} \mathcal{J}(s_1, s_2)$$



Sparse foreground



Composite model

[8] **Jarret A** et al. "A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization", *32nd European Signal Processing Conference (EUSIPCO)*, 2024

Our Composite Representer Theorem

$$\underset{s_1, s_2 \in \mathcal{M}(\mathcal{X}) \times \mathrm{L}_2(\mathcal{X})}{\arg\min} \ \frac{1}{2}\|\mathbf{y} - \mathbf{\Phi}(s_1 + s_2)\|_2^2 + \lambda_1 \|s_1\|_{\mathcal{M}} + \frac{\lambda_2}{2}\|s_2\|_{\mathrm{L}_2}^2$$

Our Representer Theorem

[Theorem 6.1]

$$\begin{cases} \widehat{s_1} \in \underset{s_1 \in \mathcal{B}}{\arg\min} \ \frac{1}{2}\|\mathrm{M}_{\lambda_2}^{-\frac{1}{2}}(\mathbf{y} - \mathbf{\Phi}(s_1))\|_2^2 + \lambda_1 \|s_1\|_{\mathcal{B}} \\ \widehat{s_2} = \frac{1}{\lambda_2}\mathbf{\Phi}^* \mathrm{M}_{\lambda_2}^{-1}(\mathbf{y} - \mathbf{w}) \end{cases}$$

$$\mathrm{M}_{\lambda_2} := \frac{1}{\lambda_2}(\mathbf{\Phi}\mathbf{\Phi}^* + \lambda_2 \mathbf{I}_L)$$

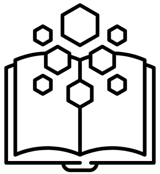
$$\mathbf{w} = \mathbf{\Phi}(\widehat{s_1})$$

[8] **Jarret A** et al. "A Decoupled Approach for Composite Sparse-Plus-Smooth Penalized Optimization", *32nd European Signal Processing Conference (EUSIPCO)*, 2024

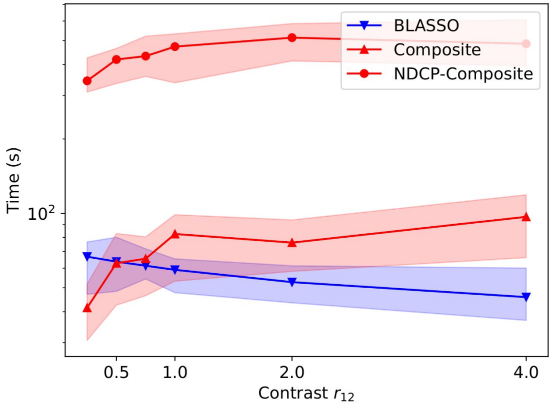
Summary and Benefits

- Better understanding of the composite optimization problem

- Nice mathematical framework



- More efficient optimization algorithms



A Short Summary of this Thesis

1. Polyatomic Frank-Wolfe
2. PolyCLEAN
3. Continuous-Domain PFW
4. Composite Reconstruction

Better understanding of the problems

Develop more efficient numerical methods

Swiss National Science Foundation

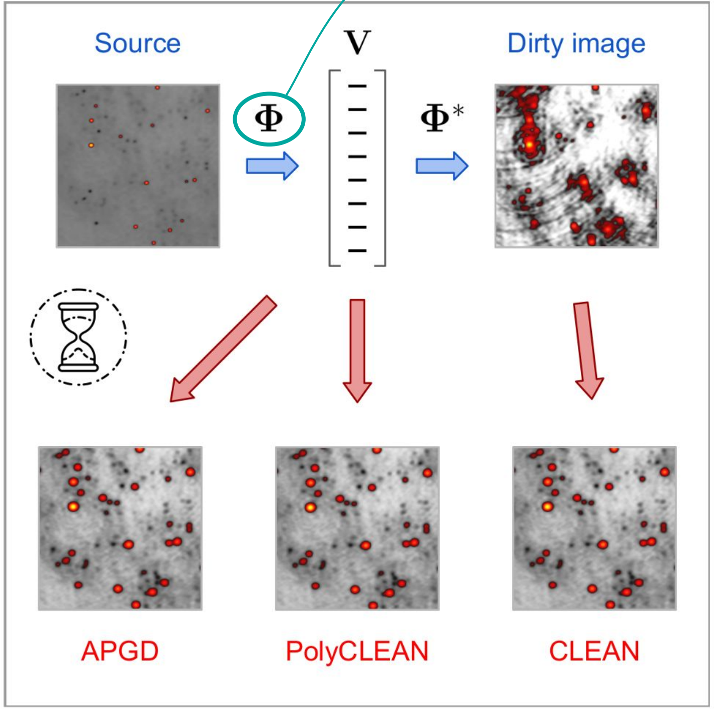
Thank you !

EPFL

LCAV

Supplementary slides

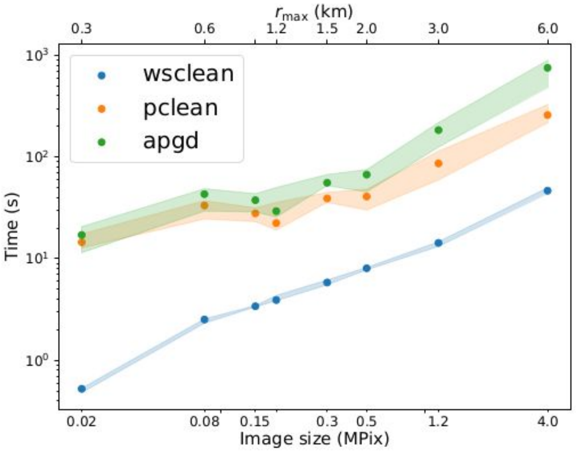
A Fast Reconstruction Method



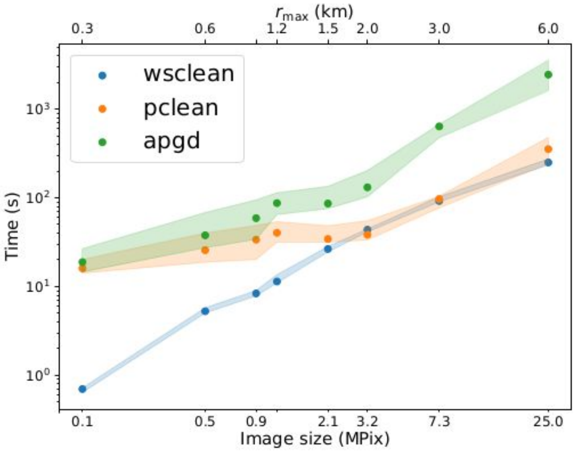
Varying number of baselines

[11] **Jarret A** et al., "PolyCLEAN: Atomic optimization for super-resolution imaging and uncertainty estimation in radio interferometry", *Astronomy & Astrophysics*, 2025.

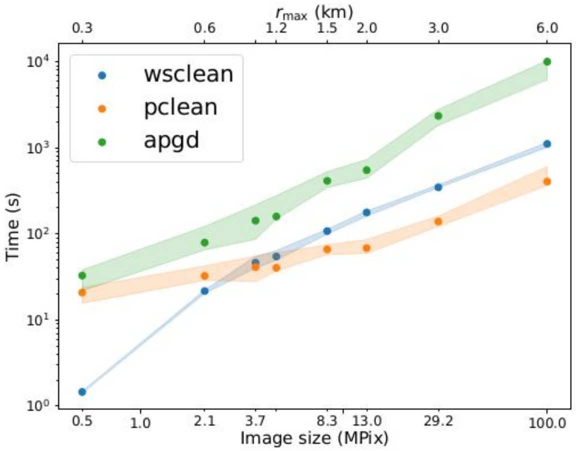
A Fast Reconstruction Method



SRF 2



SRF 5



SRF 10

[11] **Jarret A** et al., "PolyCLEAN: Atomic optimization for super-resolution imaging and uncertainty estimation in radio interferometry", *Astronomy & Astrophysics*, 2025.

Benefits of Polyatomic Frank-Wolfe

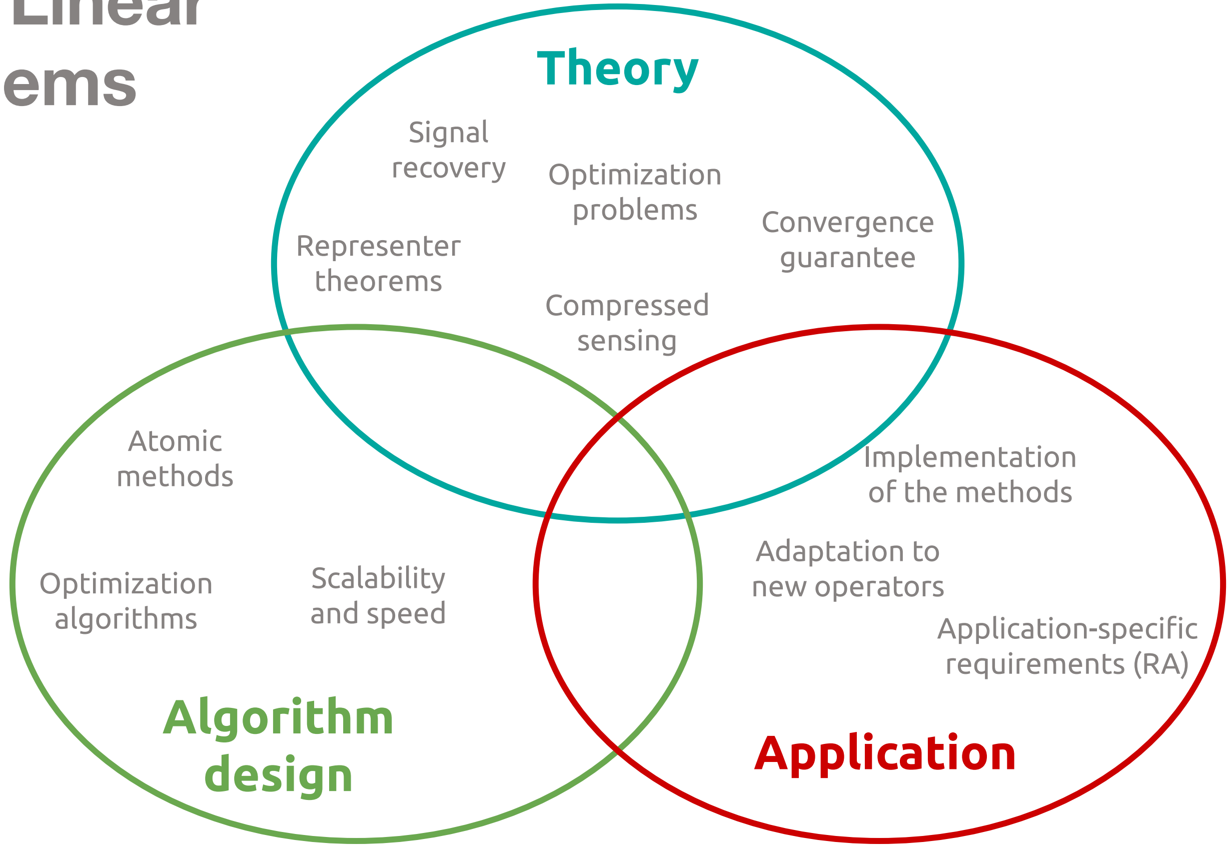
- Polyatomic $\longrightarrow$ **Fast** $\quad \mathcal{S}_k \leftarrow \mathcal{S}_{k-1} \cup \mathcal{I}_k$
- Sparse iterates $\longrightarrow$ **Scalable** $\quad \mathbf{x}_k = \sum_{i=1}^{N_k} \alpha_i^{[k]} \mathbf{e}_i^{[k]}$
- Convergence $\longrightarrow$ **Principled**

Theorem 3.2 (Convergence of Polyatomic FW).[9]

$$\mathcal{J}(\mathbf{x}_k) - \mathcal{J}^* \leq \frac{2}{k+2}(C_f + 2\delta)$$

[9] **Jarret A**, Fageot J, Simeoni M. "A Fast and Scalable Polyatomic Frank-Wolfe Algorithm for the LASSO", *IEEE Signal Processing Letters*, 2022.

Landscape of **Linear Inverse Problems**



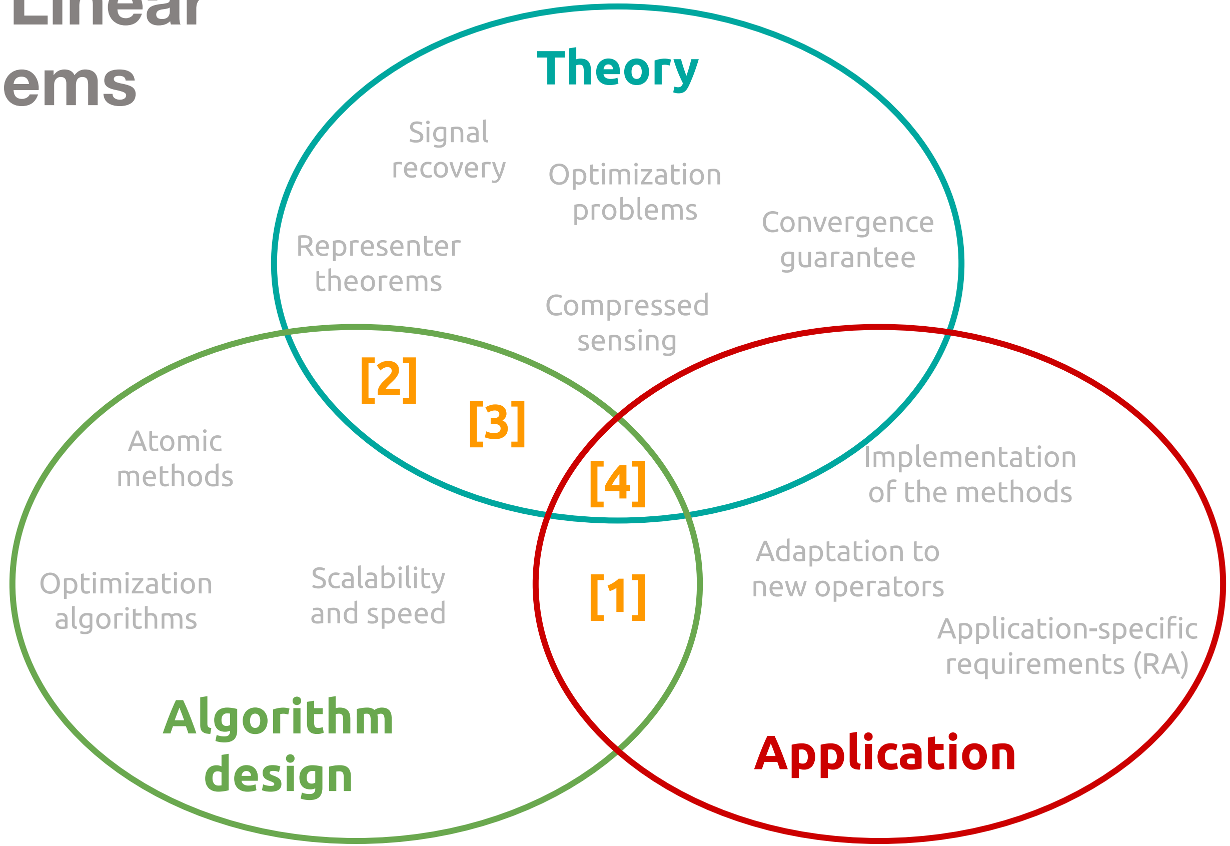
Landscape of **Linear Inverse Problems**

[1] PFW

[2] CD-PFW

[3] Composite

[4] PolyCLEAN



Conclusion and perspectives

Mathematics-aware numerical solvers:

- Principled (poly)atomic methods
- Decoupled algorithms
- Sparsity-aware processing

Chapters 3, 5

Chapter 6

Chapter 8

Open questions:

- Resolution of the reconstruction and quantitative imaging
- More advanced *traditional* methods
- Mixed learning-based approaches

Composite Representer Theorem (in the literature)

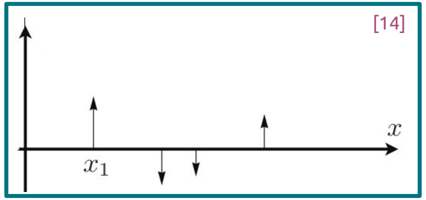
[13]

$$\underset{s_1, s_2 \in \mathcal{M}(\mathcal{X}) \times \mathrm{L}_2(\mathcal{X})}{\arg\min} \quad \frac{1}{2}\|\mathbf{y} - \mathbf{\Phi}(s_1 + s_2)\|_2^2 + \lambda_1 \|s_1\|_{\mathcal{M}} + \frac{\lambda_2}{2}\|s_2\|_{\mathrm{L}_2}^2$$

Representer theorem[13] **:**

- s_1^* $\longrightarrow$ Sparse **measure**

$$s_1^* = m[\mathbf{a}^*, \mathbf{x}^*]$$



- s_2^* $\longrightarrow$ Smooth quadratic solution

$$s_2^* = \mathbf{\Phi}^* \mathbf{u}^*$$

[13] Debarre T et al."Continuous-Domain Formulation of Inverse Problems for Composite Sparse-Plus-Smooth Signals", *IEEE Open Journal of Signal Processing*, 2021.
[14] Unser M et al., "Splines Are Universal Solutions of Linear Inverse Problems with Generalized TV Regularization", *SIAM Review*, 2017.